

# Bulgarian Mathematical Olympiad

1960 - 2008

(Only problems)

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# Bulgarian Mathematical Olympiad 1960, III Round

## *First Day*

1. Prove that the sum (and/or difference) of two irreducible fractions with different divisors cannot be an integer number. (7 points)
2. Find minimum and the maximum of the function:

$$y = \frac{x^2 + x + 1}{x^2 + 2x + 1}$$

if  $x$  can achieve all possible real values. (6 points)

3. Find  $\tan$  of the angles:  $x, y, z$  from the equations:  $\tan x : \tan y : \tan z = a : b : c$  if it is known that  $x + y + z = 180^\circ$  and  $a, b, c$  are positive numbers. (7 points)

## *Second day*

4. There are given two externally tangent circles with radii  $R$  and  $r$ .
  - (a) prove that the quadrilateral with sides - two external tangents to and chords, connecting tangents of these tangents is a trapezium;
  - (b) Find the bases and the height of the trapezium.(6 points)
5. The rays  $a, b, c$  have common starting point and doesn't lie in the same plane. The angles  $\alpha = \angle(b, c), \beta = \angle(c, a), \gamma = \angle(a, b)$ , are acute and their dimensions are given in the drawing plane. Construct with a ruler and a compass the angle between the ray  $a$  and the plane, passing through the rays  $b$  and  $c$ . (8 points)

6. In a cone is inscribed a sphere. Then it is inscribed another sphere tangent to the first sphere and tangent to the cone (not tangent to the base). Then it is inscribed third sphere tangent to the second sphere and tangent to the cone (not tangent to the base). Find the sum of the surfaces of all inscribed spheres if the cone's height is equal to  $h$  and the angle through a vertex of the cone formed by a intersection passing from the height is equal to  $\alpha$ . (6 points)

# Bulgarian Mathematical Olympiad 1961, III Round

## *First Day*

1. Let  $a$  and  $b$  are two numbers with greater common divisor equal to 1. Prove that that from all prime numbers which square don't divide the number:  $a + b$  only the square of 3 can divide simultaneously the numbers  $(a + b)^2$  and  $a^3 + b^3$ . (7 points)

2. What relation should be between  $p$  and  $q$  so that the equation

$$x^4 + px^2 + q = 0$$

have four real solutions forming an arithmetic progression? (6 points)

3. Express as a multiple the following expression:

$$A = \sqrt{1 + \sin x} - \sqrt{1 - \sin x}$$

if  $-\frac{7\pi}{2} \leq x \leq -\frac{5\pi}{2}$  and the square roots are arithmetic. (7 points)

## *Second day*

4. In a circle  $k$  are drawn the diameter  $CD$  and from the same half line of  $CD$  are chosen two points  $A$  and  $B$ . Construct a point  $S$  on the circle from the other half plane of  $CD$  such that the segment on  $CD$ , defined from the intersecting point  $M$  and  $N$  on lines  $SA$  and  $SB$  with  $CD$  to have a length  $a$ . (7 points)
5. In a given sphere with radii  $R$  are situated (inscribed) six same spheres in such a way that each sphere is tangent to the given sphere and to four of the inscribed spheres. Find the radii of inscribed spheres. (7 points)

6. Through the point  $H$ , not lying in the base of a given regular pyramid is drawn a perpendicular to the plane of the base. Prove that the sum from the segments from  $H$  to intersecting points of the perpendicular given to the planes of all non-base sides of the pyramid doesn't depend on the position of  $H$  on the base plane. (6 points)

# Bulgarian Mathematical Olympiad 1962, III Round

## *First Day*

1. It is given the sequence:  $1, 1, 2, 3, 5, 8, 13, \dots$ , each term of which after the second is equal to the sum of two terms before it. Prove that the absolute value of the difference between the square of each term from the sequence and multiple of the term before it and the term after it is equal to 1. (7 points)
2. Find the solutions of the inequality:

$$\sqrt{x^2 - 3x + 2} > x - 4$$

(7 points)

3. For which triangles the following equality is true:

$$\cos^2 \alpha \cot \beta = \cot \alpha \cos^2 \beta$$

(6 points)

## *Second day*

4. It is given the angle  $\angle XOY = 120^\circ$  with angle bisector  $OT$ . From the random point  $M$  chosen in the angle  $\angle TOY$  are drawn perpendiculars  $MC$ ,  $MA$  and  $MB$  respectively to  $OX$ ,  $OY$  and  $OT$ . Prove that:

- (a) triangle  $ABC$  is equilateral;
- (b) the following relation is true:  $MC = MA + MB$ ;
- (c) the surface of the triangle  $ABC$  is  $S = \frac{\sqrt{3}}{4} (a^2 + ab + b^2)$ , where  $MA = a$ ,  $MB = b$ .

(7 points)

5. On the base of isosceles triangle  $ABC$  is chose a random point  $M$ . Through  $M$  are drawn lines parallel to the non-base sides, intersecting  $AC$  and  $BC$  respectively at the points  $D$  and  $E$ :

- (a) prove that:  $CM^2 = AC^2 - AM \cdot BM$ ;
- (b) find the locus of the feet to perpendiculars drawn from the centre of the circumcircle over the triangle  $ABC$  to diagonals  $MC$  and  $ED$  of the parallelogram  $MECD$  when  $M$  is moving over the base  $AB$ ;
- (c) prove that :  $CM^2 = AC^2 - AM \cdot BM$  if  $M$  is over the extension of the base  $AB$  of the triangle  $ABC$ .

(7 points)

6. What is the distance from the centre of a sphere with radii  $R$  for which a plane must be drawn in such a way that the full surface of the pyramid with vertex same as the centre of the sphere and base square which is inscribed in the circle formed from intersection of the sphere and the plane is  $4 \text{ m}^2$ .

(6 points)

## Bulgarian Mathematical Olympiad 1962, IV Round

1. It is given the expression  $y = \frac{x^2-2x+1}{x^2-2x+2}$ , where  $x$  is a variable. Prove that:
- (a) if  $x_1$  and  $x_2$  are two random values of  $x$ , and  $y_1$  and  $y_2$  are the respective values of  $y$  if  $\leq x_1 < x_2, y_1 < y_2$ ;
  - (b) when  $x$  is varying  $y$  attains all possible values for which:  $0 \leq y < 1$

(5 points)

2. It is given a circle with center  $O$  and radii  $r$ .  $AB$  and  $MN$  are two random diameters. The lines  $MB$  and  $NB$  intersects tangent to the circle at the point  $A$  respectively at the points  $M'$  and  $N'$ .  $M''$  and  $N''$  are the middlepoints of the segments  $AM'$  and  $AN'$ . Prove that:

- (a) around the quadrilateral  $MNN'M'$  may be circumscribed a circle;

- (b) the heights of the triangle  $M''N''B$  intersects in the middle-point of the radii  $OA$ .

(5 points)

3. It is given a cube with sidelength  $a$ . Find the surface of the intersection of the cube with a plane, perpendicular to one of its diagonals and which distance from the centre of the cube is equal to  $h$ .

(4 points)

4. There are given a triangle and some its internal point  $P$ .  $x, y, z$  are distances from  $P$  to the vertices  $A, B$  and  $C$ .  $p, q, r$  are distances from  $P$  to the sides  $BC, CA, AB$  respectively. Prove that:

$$xyz = (q + r)(r + p)(p + q)$$

(6 points)



# Bulgarian Mathematical Olympiad 1963, III Round

## *First Day*

1. From the three different digits  $x, y, z$  are constructed all possible three-digit numbers. The sum of these numbers is 3 times bigger than the number which all three digits are equal to  $x$ . Find the numbers:  $x, y, z$ .

(7 points)

2. Solve the inequality:

$$\frac{1}{2(x-1)} - \frac{4}{x} + \frac{15}{2(x+1)} \geq 1$$

(7 points)

3. If  $\alpha, \beta, \gamma$  are the angles of some triangle prove the equality:

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + 2 \cos \alpha \cos \beta + \cos \gamma = 1$$

(6 points)

## *Second day*

4. Construct a triangle, similar to a given triangle one if one of its vertices is same as a point given in advance and the other two vertices lie at a given in advance circle. (HINT: You may use circumscribed around required triangle circle) (8 points)

5. A regular tetrahedron is cut from a plane parallel to some of its base edges and to some of the other non-base edges, non intersecting the given base line. Prove that:

- (a) the intersection is a rectangle;
- (b) perimeter of the intersection doesn't depend of the situation of the cutting plane.

(5 points)

6. Find dihedral line  $\varphi$ , between base wall and non-base wall of regular pyramid which base is quadrilateral if it is known that the radii of the circumscribed sphere bigger than the radii of the inscribed sphere.

(7 points)

## Bulgarian Mathematical Olympiad 1963, IV Round

1. Find all three-digit numbers which remainders after division by 11 give quotient, equal to the sum of it's digits squares. (4 points)

2. It is given the equation  $x^2 + px + 1 = 0$ , with roots  $x_1$  and  $x_2$ ;

- (a) find a second-degree equation with roots  $y_1, y_2$  satisfying the conditions:  $y_1 = x_1(1 - x_1)$ ,  $y_2 = x_2(1 - x_2)$ ;  
(b) find all possible values of the real parameter  $p$  such that the roots of the new equation lies between -2 and 1.

(5 points)

3. In the trapezium  $ABCD$  with on the non-base segment  $AB$  is chosen a random point  $M$ . Through the points  $M, A, D$  and  $M, B, C$  are drawn circles  $k_1$  and  $k_2$  with centers  $O_1$  and  $O_2$ . Prove that:

- (a) the second intersection point  $N$  of  $k_1$  and  $k_2$  lies on the other non-base segment  $CD$  or on its continuation;  
(b) the length of the line  $O_1O_2$  doesn't depend of the situation on  $M$  over  $AB$ ;  
(c) the triangles  $O_1MO_2$  and  $DMC$  are similar. Find such a position of  $M$  over  $AB$  that makes  $k_1$  and  $k_2$  with the same radii.

(6 points)

4. In the tetrahedron  $ABCD$  three of the sides are right-angled triangles and the second in not an obtuse triangle. Prove that:

- (a) the fourth wall of the tetrahedron is right-angled triangle if and only if exactly two of the plane angles having common vertex with the some of vertices of the tetrahedron are equal.
- (b) when all four walls of the tetrahedron are right-angled triangles its volume is equal to  $\frac{1}{6}$  multiplied by the multiple of three shortest edges not lying on the same wall.

(5 points)

REMARK FOR (B) - more correct statement should be: *... its volume is equal to  $\frac{1}{6}$  multiplied by the multiple of two shortest edges and an edge not lying on the same wall.*

## Bulgarian Mathematical Olympiad 1964, III Round

### *First Day*

1. Find four-digit number:  $\overline{xyzt}$  which is an exact cube of natural number if its four digits are different and satisfy the equations:  
 $2x = y - z$  and  $y = t^2$ . (7 points)

2. Find all possible real values of  $k$  for which roots of the equation

$$(k + 1)x^2 - 3kx + 4k = 0$$

are real and each of them is greater than -1. (7 points)

3. Find all real solutions of the equation:

$$x^2 + 2x \cos(xy) + 1 = 0$$

(7 points)

### *Second day*

4. A circle  $k$  and a line  $t$  are tangent at the point  $T$ . Let  $M$  is a random point from  $t$  and  $MA$  is the second tangent to  $k$ . There are drawn a diameter  $AB$  and a perpendicular  $TC$  to  $AB$  ( $C$  lies on  $AB$ ):

- (a) prove that the intersecting point  $P$  of the lines  $MB$  and  $TC$  is a midpoint of the segment  $TC$ ;  
 (b) find the locus of  $P$  when  $M$  is moving over the line  $t$ .

(7 points)

5. In the tetrahedron  $ABCD$  all pair of opposite edges are equal. Prove that the lines passing through their midpoints are mutually perpendicular and are axis of symmetry of the given tetrahedron. (7 points)

6. Construct a right-angled triangle by given hypotenuse  $c$  and an obtuse angle  $\varphi$  between two medians to the cathets. Find the allowed range in which the angle  $\varphi$  belongs (min and max possible value of  $\varphi$ ).

## Bulgarian Mathematical Olympiad 1964, IV Round

1. A  $6n$ -digit number is divisible by 7. Prove that if its last digit is moved at the beginning of the number (first position) then the new number is also divisible by 7. (5 points)

2. Find all possible  $n$ -tuples of reals:  $x_1, x_2, \dots, x_n$  satisfying the system:

$$\begin{cases} x_1 \cdot x_2 \cdots x_n = 1 \\ x_1 - x_2 \cdot x_3 \cdots x_n = 1 \\ x_1 \cdot x_2 - x_3 \cdot x_4 \cdots x_n = 1 \\ \dots \\ x_1 \cdot x_2 \cdots x_{n-1} - x_n = 1 \end{cases}$$

(4 points)

3. There are given two intersecting lines  $g_1, g_2$  and a point  $P$  in their plane such that  $\angle(g_1, g_2) \neq 90^\circ$ . Its symmetrical points on any random point  $M$  in the same plane with respect to the given planes are  $M_1$  and  $M_2$ . Prove that:

- (a) the locus of the point  $M$  for which the point  $M_1, M_2$  and  $P$  lies on a common line is a circle  $k$  passing intersecting point of  $g_1$  and  $g_2$ .
- (b) the point  $P$  an orthocenter of the triangle, inscribed in the circle  $k$  sides of which lies at the lines  $g_1$  and  $g_2$ .

(6 points)

4. Let  $a_1, b_1, c_1$  are three lines each two of them are mutually crossed and aren't parallel to some plane. The lines  $a_2, b_2, c_2$  intersects

the lines  $a_1, b_1, c_1$  at the points  $a_2$  in  $A, C_2, B_1$ ;  $b_2$  in  $C_1, B, A_2$ ;  $c_2$  in  $B_2, A_1, C$  respectively in such a way that  $A$  is the middle line of  $B_1C_2$ ,  $B$  is the middle of  $C_1A_2$  and  $C$  is the middle of  $A_1B_2$ . Prove that:

- (a)  $A$  is the middle of the  $B_2C_1$ ,  $B$  is the middle of  $C_2A_1$  and  $C$  is the middle of  $A_2B_1$ ;
- (b) triangles  $A_1B_1C_1$  and  $A_2B_2C_2$  are the same. ( $A_1B_1C_1A_2B_2C_2$  - is a prism).

(5 points)

# Bulgarian Mathematical Olympiad 1965, III Round

## *First Day*

1. On a circumference are written 1965 digits, It is known if we read the digits on the same direction as the clock hand is moving, resulting 1965-digit number will be divisible to 27. Prove that if we start reading of the digits from some other position the resulting 1965-digit number will be also divisible to 27. (7 points)

2. Find all real roots of the equation:

$$\sqrt{x^2 - 2p} + \sqrt{4x^2 - p - 2} = x$$

where  $p$  is real parameter. (points)

3. Prove that if  $\alpha, \beta, \gamma$  are angles of some triangle then

$$A = \cos \alpha + \cos \beta + \cos \gamma < 2$$

(6 points)

## *Second day*

4. It is given an acute-angled triangle  $ABC$ . Perpendiculars to  $AC$  and  $BC$  drawn from the points  $A$  and  $B$  intersects in the point  $P$ .  $Q$  is the projection of  $P$  on  $AB$ . Prove that the arms of  $\angle ACB$  cut from a line passing through  $Q$  and different from  $AB$  segment bigger than the segment  $AB$ . (7 Points)
5. Construct a triangle  $ABC$  by given side  $AB = c$  and distances  $p$  and  $q$  from vertices  $A$  and  $B$  to the angle bisector of angle  $C$ . Express the area of the triangle  $ABC$  by  $c, p$  and  $q$ . (7 points)
6. Let  $P$  is not an external point to the tetrahedron  $DABC$  different from the point  $D$ . Prove that from the segments  $PA, PB, PC$  can be chosen a segment that is shorter from some of the segments  $DA, DB, DC$ .

(6 points)

## Bulgarian Mathematical Olympiad 1965, IV Round

- The numbers 2, 3, 7 have the property that the multiple of any two of them increased by 1 is divisible of the third number. Prove that this triple of integer numbers greater than 1 is the only triple with the given property. (6 points)
- Prove the inequality:

$$(1 + \sin^2 \alpha)^n + (1 + \cos^2 \alpha)^n \geq 2 \left(\frac{3}{2}\right)^n$$

is true for every natural number  $n$ . When does equality holds? (5 points)

- In the triangle  $ABC$  angle bisector  $CD$  intersects circumscribed around  $ABC$  circle at the point  $K$ .
  - Prove the equalities:

$$\frac{1}{JD} - \frac{1}{JK} = \frac{1}{CJ} \quad , \quad \frac{CJ}{JD} - \frac{JD}{DK} = 1$$

where  $J$  is the centre of the inscribed circle.

- On the segment  $CK$  is chosen a random point  $P$  with projections on  $AC$ ,  $BC$ ,  $AB$  respectively:  $P_1$ ,  $P_2$ ,  $P_3$ . The lines  $PP_3$  and  $P_1P_2$  intersects at a point  $M$ . Find the locus of  $M$  when  $P$  is moving around the  $CK$  segment.

(9 points)

- In the space are given crossed lines  $s$  and  $t$  such that  $\angle(s, t) = 60^\circ$  and a segment  $AB$  perpendicular to them. On  $AB$  is chosen a point  $C$  for which  $AC : CB = 2 : 1$  and the points  $M$  and  $N$  are moving on the lines  $s$  and  $t$  in such a way that  $AM = 2BN$ . Prove that<sup>1</sup>:

- the segment  $MN$  is perpendicular to  $t$ ;

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<sup>1</sup>In the statement should be said that vectors  $\overrightarrow{AM}$  and  $\overrightarrow{BN}$  have the angle between them  $60^\circ$



- (b) the plane  $\alpha$ , perpendicular to  $AB$  in point  $C$  intersects the plane  $CMN$  on fixed line  $\ell$  with given direction in respect to  $s$  and  $t$ ;
- (c) reverse, all planes passing by  $ell$  and perpendicular to  $AB$  intersects the lines  $s$  and  $t$  respectively at points  $M$  and  $N$  for which  $AM = 2BN$  and  $MN \perp t$ .

(6 points)

# Bulgarian Mathematical Olympiad 1966, III Round

## *First Day*

1. Find all possible values of the natural number  $n$  for which the number  $n^{n+1} - (n+1)^n$  is divisible by 3. (6 Points)
2. Prove the inequality:

$$\log_{b+c} a^2 + \log_{c+a} b^2 + \log_{a+b} c^2 \geq 3$$

where the numbers  $a, b, c$  are not smaller than 2. (8 points)

3. In the plane are given  $n$  points. It is known that if we choose any four of these points there are three points that lie on a common straight line. Prove that all  $n$  points maybe except one lie on a common straight line. (6 points)

## *Second day*

4. It is given a tetrahedron  $ABCD$ . Medians of the triangle  $BCD$  meets each other in point  $M$ . Prove the inequality:

$$AM \leq \frac{AB + AC + AD}{3}$$

(5 Points)

5. In the triangle  $ABC$  the angle bisector, median and the height drawn respectively through the vertices  $A, B, C$  intersects at a common point. Prove that the angles  $A, B, C$  satisfies the equation:

$$\tan A = \frac{\sin B}{\cos C}$$

(9 Points)

6. In a tourist tour participates yang people, girls and boys. It is known that every boy knows at least one girl but he doesn't know all the girls, and every girl knows at least one of the boys but she doesn't know all the boys. Prove that from participants may be chosen two boys and two girls such that each of the selected boys knows one of the selected girls but doesn't know the other selected girl and each of the selected girls knows one of the selected boys but doesn't know the other selected boy. (6 Points)

## Bulgarian Mathematical Olympiad 1966, IV Round

1. Prove that the equality:

$$3x(x - 3y) = y^2 + z^2$$

doesn't have other integer solutions except  $x = 0, y = 0, z = 0$ .  
(5 points)

2. Prove that for every four positive numbers  $a, b, c, d$  is true the following inequality:

$$\sqrt{\frac{a^2 + b^2 + c^2 + d^2}{4}} \geq \sqrt[3]{\frac{abc + abd + acd + bcd}{4}}$$

(7 points)

3. (a) In the plane of the triangle  $ABC$  find a point with the following property: its symmetrical points with respect to middle points of the sides of the triangle lies on the circumscribed circle.
- (b) Construct the triangle  $ABC$  if it is known the positions of the orthocenter  $H$ , middle point of the side  $AB$  and the middle point of the segment joining the foots of the heights through vertices  $A$  and  $B$ .

(9 points)

4. It is given a tetrahedron with vertices  $A, B,$

- (a) Prove that there exists vertex of tetrahedron with the following property: the three edges of that tetrahedron can be constructed a triangle.
- (b) Over the edges  $DA$ ,  $DB$  and  $DC$  are given the points  $M$ ,  $N$  and  $P$  for which:

$$DM = \frac{DA}{n}, \quad DN = \frac{DB}{n+1}, \quad DP = \frac{DC}{n+2}$$

where  $n$  is a natural number. The plane defined by the points  $M$ ,  $N$  and  $P$  is  $\alpha_n$ . Prove that all planes  $\alpha_n$ , ( $n = 1, 2, 3, \dots$ ) passes through a single straight line.

(9 points)

## Bulgarian Mathematical Olympiad 1967, III Round

### *First Day*

1. Find four digit number which on division by 139 gives a remainder 21 and on division by 140 gives a remainder 7. (7 Points)
2. There are given 12 numbers  $a_1, a_2, \dots, a_{12}$  satisfying the conditions:

$$a_2(a_1 - a_2 + a_3) < 0$$

$$a_3(a_2 - a_3 + a_4) < 0$$

...

$$a_{11}(a_{10} - a_{11} + a_{12}) < 0$$

Prove that among these numbers there are at least three positive and three negative. (6 points)

3. On time of suspension of arms around round (circular) table are situated few knights from two enemy's camps. It is known that the count of knights with an enemy on its right side is equal to the count of knights with a friend on its right side. Prove that the total count of the knights situated around the circular table is divisible by 4. (7 points)

### *Second day*

4. In the triangle  $ABC$  from the foot of the altitude  $CD$  is drawn a perpendicular  $DE$  to the side  $BC$ . On the line  $DC$  is taken point  $H$  for which  $DH : HE = DB : DA$ . Prove that the segments  $CH$  and  $AE$  are mutually perpendicular. (6 Points)
5. Prove that for each acute angled triangle is true the following inequality:

$$m_a + m_b + m_c \leq 4R + r$$

(8 Points)

6. From the tetrahedrons  $ABCD$  with a given volume  $V$  for which:

$$AC \perp CD \perp DB \perp AC$$

find this one with the smallest radii of the circumscribed sphere.  
(6 Points)

## Bulgarian Mathematical Olympiad 1967, IV Round

1. The numbers 12, 14, 37, 65 are one of the solutions of the equation:

$$xy - xz + yt = 182$$

What number of what letter corresponds? (5 points)

2. Prove that:

- (a) if  $y < \frac{1}{2}$  and  $n \geq 3$  is a natural number then:  $(y + 1)^n \geq y^n + (1 + 2y)^{\frac{n}{2}}$ ;  
 (b) if  $x, y, z$  and  $n \geq 3$  are natural numbers for which:  $x^2 - 1 \leq 2y$  then  $x^n + y^n \neq z^n$ .

(9 points)

3. It is given a right-angled triangle  $ABC$  and its circumcircle  $k$ .

- (a) prove that the radii of the circle  $k_1$  tangent to the cathets of the triangle and to the circle  $k$  is equal to the diameter of the incircle of the triangle  $ABC$ .  
 (b) on the circle  $k$  may be found a point  $M$  for which the sum  $MA + MB + MC$  is biggest possible.

(11 points)

4. Outside of the plane of the triangle  $ABC$  is given point  $D$ .

- (a) prove that if the segment  $DA$  is perpendicular to the plane  $ABC$  then orthogonal projection of the orthocenter of the triangle  $ABC$  on the plane  $BCD$  coincides with the orthocenter of the triangle  $BCD$ .

- (b) for all tetrahedrons  $ABCD$  with base, the triangle  $ABC$  with smallest of the four heights that from the vertex  $D$ , find the locus of the foot of that height.

(10 points)

## Bulgarian Mathematical Olympiad 1968, III Round

### *First Day*

1. Find four digit number  $\overline{1xyz}$ , if two of the numbers  $\overline{xz}$ ,  $\overline{yx} + 1$ ,  $\overline{zy} - 2$  are divisible by 7 and  $x + 2y + z = 29$ . (6 Points)
2. Find the numbers  $A, B, C$  in such a way that for every natural number  $n$  is true the following equality

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = \frac{An + B}{2^n} + C$$

(7 Points)

3. Solve the inequality

$$(1 - \cos x)(1 + \cos 2x)(1 - \cos 3x) < \frac{1}{2}$$

(7 Points)

### *Second day*

4. The points  $A, B, C$  and  $D$  are sequential vertices of regular polygon and the following equality is satisfied

$$\frac{1}{AB} = \frac{1}{AC} + \frac{1}{AD}$$

How much sides the polygon have ? (6 Points)

5. In a triangle  $ABC$  over the median  $CM$  is chosen a random point  $O$ . The lines  $AO$  and  $BO$  intersects the sides  $BC$  and  $AC$  at the points  $K$  and  $L$  respectively. Prove that if  $AC > BC$  then  $AK > BL$ . (6 Points)

6. The base of pyramid  $SABCD$  (with base  $ABCD$ ) is a quadrilateral with mutually perpendicular diagonals. The orthogonal projection of the vertex  $S$  over the base of the pyramid coincides with the intersection point of the diagonals  $AC$  and  $BD$ . Prove that the orthogonal projections of the point  $O$  over the walls of the pyramid lies over a common circle. (8 Points)



# Bulgarian Mathematical Olympiad 1968, IV Round

## *First Day*

1. Find all possible natural values of  $k$  for which the system

$$\begin{cases} x_1 + x_2 + \cdots + x_k = 9 \\ \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} = 1 \end{cases}$$

have solutions in positive numbers. Find these solutions.

(6 points, I. Dimovski)

2. Find all functions  $f(x)$ , defined for every  $x, y$  satisfying the equality

$$xf(y) + yf(x) = (x + y)f(x)f(y)$$

for every  $x, y$ . Prove that exactly two of them are continuous.

(6 points, I. Dimovski)

3. Prove that a binomial coefficient  $\binom{n}{k}$  is odd if and only if all digits 1 of  $k$ , when  $k$  is written in binary digit system are on the same positions when  $n$  is written in binary system. (8 points, I. Dimovski)

## *Second day*

4. Over the line  $g$  are given the segment  $AB$  and a point  $C$  external for  $AB$ . Prove that over  $g$  there exists at least one pair of points  $P, Q$  symmetrical with respect to  $C$ , which divide the segment  $AB$  internally and externally in the same ratios, i.e.

$$\frac{PA}{PB} = \frac{QA}{QB} \quad (1)$$

Opposite if  $A, B, P, Q$  are such points from the line  $g$  for which is satisfied (1), prove that the middle point  $C$  of the segment  $PQ$  is external point for the segment  $AB$ . (6 points, K. Petrov)

5. The point  $M$  is internal for the tetrahedron  $ABCD$  and the intersection points of the lines  $AM$ ,  $BM$ ,  $CM$  and  $DM$  with the opposite walls are denoted with  $A_1$ ,  $B_1$ ,  $C_1$ ,  $D_1$  respectively. It is given also that the ratios  $\frac{MA}{MA_1}$ ,  $\frac{MB}{MB_1}$ ,  $\frac{MC}{MC_1}$  and  $\frac{MD}{MD_1}$  are equal to the same number  $k$ . Find all possible values of  $k$ . (8 points, K. Petrov)
6. Find the kind of the triangle if

$$\frac{a \cos \alpha + b \cos \beta + c \cos \gamma}{a \sin \alpha + b \sin \beta + c \sin \gamma} = \frac{2p}{9R}$$

( $\alpha$ ,  $\beta$ ,  $\gamma$  are the measures of the angles,  $a$ ,  $b$ ,  $c$ ,  $p$ ,  $R$  are the lengths of the sides, the  $p$ -semiperimeter, the radii of the circumcircle of the triangle).

(6 points, K. Petrov)

## Bulgarian Mathematical Olympiad 1969, III Round

### *First Day*

1. Prove that for every natural number  $n$  the number  $N = 1 + 2^{2 \cdot 5^n}$  is divisible by  $5^{n+1}$ . (6 Points)
2. Prove that the polynomial  $f(x) = x^5 - x + a$ , where  $a$  is an integer number which is not divisible by 5, cannot be written as a product of two polynomials with lower degree. (8 Points)
3. There are given 20 different natural numbers smaller than 70. Prove that among their differences there are two equals. (6 Points)

### *Second day*

4. It is given acute-angled triangle with sides  $a, b, c$ . Let  $p, r$  and  $R$  are semiperimeter, radii of inscribed and radii of circumscribed circles respectively. It's center of gravity is also a midpoint of the segment with edges incenter and circumcenter. Prove that the following equality is true:

$$7(a^2 + b^2 + c^2) = 12p^2 + 9R(R - 6r)$$

(7 Points)

5. In the triangle pyramid  $OABC$  with base  $ABC$ , the edges  $OA, OB, OC$  are mutually perpendicular (each two of them are perpendicular).
  - (a) From the center of circumscribed sphere around the pyramid is drawn a plane, parallel to the wall  $ABC$ , which intersects the edges  $OA, OB$  and  $OC$  respectively in the points  $A_1, B_1, C_1$ . Find the ratio between the volumes of the pyramids  $OABC$  and  $OA_1B_1C_1$ .
  - (b) Prove that if the walls  $OBC, OAC$  and  $OAB$  have the angles with the base  $ABC$  respectively  $\alpha, \beta$  and  $\gamma$  then

$$\frac{h - r}{r} = \cos \alpha + \cos \beta + \cos \gamma$$

where  $h$  is the distance between  $O$  and  $ABC$  plane and  $r$  is the radii of the inscribed in the pyramid  $OABC$  sphere.

(8 Points)

6. Prove the equality

$$1 + \frac{\cos x}{\cos^1 x} + \frac{\cos 2x}{\cos^2 x} + \cdots + \frac{\cos nx}{\cos^n x} = \frac{\sin(n+1)x}{\sin x \cos^n x}$$

if  $\cos x \neq 0$  and  $\sin x \neq 0$ . (5 Points)

## Bulgarian Mathematical Olympiad 1969, IV Round

### *First Day*

1. If the sum of  $x^5$ ,  $y^5$  and  $z^5$ , where  $x$ ,  $y$  and  $z$  are integer numbers, is divisible by 25 then the sum of some two of them is divisible by 25.
2. Prove that

$$S_n = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} < 2$$

for every  $n \in \mathbb{N}$ .

3. Some of the points in the plane are *white* and other are *blue* (every point from the plane is white or blue). Prove that for every positive number  $r$ :
  - (a) there are at least two points with different color and the distance between them is equal to  $r$ ;
  - (b) there are at least two points with the same color and the distance between them is equal to  $r$ ;
  - (c) will the statements above be true if the *plane* is replaced with the word *line*?

### *Second day*

4. Find the sides of the triangle if it is known that the inscribed circle meets one of its medians in two points and these points divide the median to three equal segments and the area of the triangle is equal to  $6\sqrt{14}$  cm<sup>2</sup>.
5. Prove the equality:

$$\prod_{k=1}^{2m} \cos \frac{k\pi}{2m+1} = \frac{(-1)^m}{4m}$$

6. It is given that  $r = [3(\sqrt{6} - 1) - 4(\sqrt{3} + 1) + 5\sqrt{2}] R$  where  $r$  and  $R$  are radii of the inscribed and circumscribed spheres in the regular  $n$ -angled pyramid. If it is known that the centers of the spheres given coincides:
- find  $n$ ;
  - if  $n = 3$  and the lengths of all edges are equal to  $a$  find the volumes of the parts from the pyramid after drawing a plane  $\mu$ , which intersects two of the edges passing through point  $A$  respectively in the points  $E$  and  $F$  in such a way that  $|AE| = p$  and  $|AF| = q$  ( $p < a$ ,  $q < a$ ), intersects the extension of the third edge behind opposite of the vertex  $A$  wall in the point  $G$  in such a way that  $|AG| = t$  ( $t > a$ ).

# Bulgarian Mathematical Olympiad 1970, III Round

## *First Day*

1. Prove the the inequality

$$\frac{1-a}{1+a} + \frac{1-b}{1+b} + \frac{1-c}{1+c} \geq \frac{3}{2}$$

where  $a \geq 0$ ,  $b \geq 0$ ,  $c \geq 0$  and  $a + b + c = 1$ . (5 Points)

2. There are given the numbers  $a = 123456789$  and  $b = 987654321$ . Find:

- (a) biggest common divisor of  $a$  and  $b$ ;
- (b) remainder after division of the smallest common multiple of  $a$  and  $b$  to 11.

(8 Points)

3. Points of plane are divided to three groups *white*, *green*, *red*. Prove that there exists at least one pair of points with the same color (from the same group), which have a distance to each other equal to 1. (7 Points)

## *Second day*

4. In the triangle  $ABC$  is given a point  $M$  and through  $M$  are drawn lines, parallel to the sides of the triangle. These lines cut from the triangle three smaller triangles in such a way that one of the vertices of each triangle is a vertex of the biggest triangle  $ABC$ . Let  $P_a$ ,  $P_b$ ,  $P_c$  are perimeters of the given triangle and  $S_a$ ,  $S_b$ ,  $S_c$  are the areas of these triangles.  $P$  and  $S$  are the perimeter and the area of thye triangle  $ABC$ . Prove that:

(a) 
$$P = \frac{P_a + P_b + P_c}{2};$$

(b) 
$$\sqrt{S} = \frac{\sqrt{S_a} + \sqrt{S_b} + \sqrt{S_c}}{2}.$$

(5 Points)

5. Calculate without using logarithmic table or other additional tools

$$S_n(\alpha) = \frac{\cos 2\alpha}{\sin 3\alpha} + \frac{\cos 6\alpha}{\sin 9\alpha} + \cdots + \frac{\cos 2 \cdot 3^{n-1}\alpha}{\sin 3^n\alpha}$$

for  $\alpha = 18^\circ$ , where  $n$  is a natural number in the form  $1 + 4k$ . (7 Points)

6. It is given quadrilateral prism  $ABCD A_1 B_1 C_1 D_1$ , for which the smallest distance between  $AA_1$  and  $BD_1$  is 8m and the distance from the vertex  $A_1$  to the plane of the triangle  $ACB_1$  is  $\frac{24}{\sqrt{13}}$ m. Through middlepoints of the edges  $AB$  and  $BC$  is constructed intersection which divides the axis of the prism in ratio 1 : 3 from bottom base ( $ABCD$ ):

- what is the shape of the intersection;
- calculate the area of the intersection.

(8 Points)

## Bulgarian Mathematical Olympiad 1970, IV Round

### *First Day*

1. Find all natural numbers  $a > 1$ , with the property: every prime divisor of  $a^6 - 1$  divides also at least one of the numbers  $a^3 - 1$ ,  $a^2 - 1$ .

(7 Points, K. Dochev)

2. Two bicyclists traveled the distance from  $A$  to  $B$ , which is 100 Km with speed 30 Km/h and it is known that the first is started 30 minutes before the second. 20 minutes after the start of the first bicyclist from  $A$  is started a control car which speed is 90 Km/h and it is known that the car is reached the first bicyclist and is driving together with him 10 minutes, went back to the second and was driving 10 minutes with him and after that the car is started again to the first bicyclist with speed 90 Km/h and etc. to the end of the distance. How many times the car were drive together with the first bicyclist? (5 Points, K. Dochev)

3. Over a chessboard (with 64 squares) are situated 32 white and 32 black pools. We say that two pools form a mixed pair when they are with different colors and lies on one and the same row or column. Find the maximum and the minimum of the mixed pairs for all possible situations of the pools.  
(8 Points, K. Dochev)

*Second day*

4. Let  $\delta_0 = \triangle A_0B_0C_0$  is a triangle with vertices  $A_0, B_0, C_0$ . Over each of the side  $B_0C_0, C_0A_0, A_0B_0$  are constructed squares in the halfplane, not containing the respective vertex  $A_0, B_0, C_0$  and  $A_1, B_1, C_1$  are the centers of the constructed squares. If we use the triangle  $\delta_1 = \triangle A_1B_1C_1$  in the same way we may construct the triangle  $\delta_2 = \triangle A_2B_2C_2$ ; from  $\delta_2 = \triangle A_2B_2C_2$  we may construct  $\delta_3 = \triangle A_3B_3C_3$  and etc. Prove that:
- (a) segments  $A_0A_1, B_0B_1, C_0C_1$  are respectively equal and perpendicular to  $B_1C_1, C_1A_1, A_1B_1$ ;
  - (b) vertices  $A_1, B_1, C_1$  of the triangle  $\delta_1$  lies respectively over the segments  $A_0A_3, B_0B_3, C_0C_3$  (defined by the vertices of  $\delta_0$  and  $\delta_1$ ) and divide them in ratio 2:1. (7 Points, K. Dochev)
5. Prove that for  $n \geq 5$  the side of regular inscribed in a circle  $n$ -gon is bigger than the side of regular circumscribed around the same circle  $n + 1$ -gon and if  $n \leq 4$  is true the opposite statement. (6 Points)
6. In the space are given the points  $A, B, C$  and a sphere with center  $O$  and radii 1. Find the point  $X$  from the sphere for which the sum  $f(X) = |XA|^2 + |XB|^2 + |XC|^2$  attains its maximal and minimal value. ( $|XA|$  is the distance from  $X$  to  $A$ ,  $|XB|$  and  $|XC|$  are defined by analogy). Prove that if the segments  $OA, OB, OC$  are mutually perpendicular and  $d$  is the distance from the center  $O$  to the center of gravity of the triangle  $ABC$  then:
- (a) the maximum of  $f(X)$  is equal to  $9d^2 + 3 + 6d$ ;
  - (b) the minimum of  $f(X)$  is equal to  $9d^2 + 3 - 6d$ .



(7 Points, K. Dochev and I. Dimovski)

# Bulgarian Mathematical Olympiad 1971, III Round

## *First Day*

1. Prove that the equation

$$x^{12} - 11y^{12} + 3z^{12} - 8t^{12} = 1971^{1970}$$

don't have solutions in integer numbers. (5 Points)

2. Solve the system:

$$(a) \begin{cases} x = \frac{2y}{1+y^2} \\ y = \frac{2z}{1+z^2} \\ z = \frac{2x}{1+x^2} \end{cases} \quad (b) \begin{cases} x = \frac{2y}{1-y^2} \\ y = \frac{2z}{1-z^2} \\ z = \frac{2x}{1-x^2} \end{cases}$$

( $x, y, z$  are real numbers). (7 Points)

3. Let  $E$  is a system of 17 segments over a straight line. Prove:

- (a) or there exist a subsystem of  $E$  that consist from 5 segments which on good satisfying ardering includes monotonically in each one (the first on the second, the second on the next and ect.)
- (b) or can be found 5 segments from  $\epsilon$ , no one of them is contained in some of the other 4.

(8 Points)

## *Second day*

4. Find all possible conditions for the real numbers  $a, b, c$  for which the equation  $a \cos x + b \sin x = c$  have two solutions,  $x'$  and  $x''$ , for which the difference  $x' - x''$  is not divisible by  $\pi$  and  $x' + x'' = 2k\pi + \alpha$  where  $\alpha$  is a given number and  $k$  is an integer number. (6 Points)

5. Prove that if in a triangle two of three angle bisectors are equal the triangle is isosceles. (6 Points)
6. It is given a cube with edge  $a$ . On distance  $\frac{a\sqrt{3}}{8}$  from the center of the cube is drawn a plane perpendicular to some of diagonals of the cube:
- (a) find the shape/kind of the intersection of the plane with the cube;
- (b) calculate the area of this intersection.

(8 Points)

## Bulgarian Mathematical Olympiad 1971, IV Round

### *First Day*

1. A natural number is called *triangled* if it may be presented in the form  $\frac{n(n+1)}{2}$ . Find all values of  $a$  ( $1 \leq a \leq 9$ ) for which there exist a *triangled* number all digit of which are equal to  $a$ .
2. Prove that the equation

$$\sqrt{2 - x^2} + \sqrt[3]{3 - x^3} = 0$$

have no real solutions.

3. There are given 20 points in the plane, no three of which lies on a single line. Prove that there exist at least 969 quadrilaterals with vertices from the given points.

### *Second day*

4. It is given a triangle  $ABC$ . Let  $R$  is the radii of the circumcircle of the triangle and  $O_1, O_2, O_3$  are the centers of external incircles of the triangle  $ABC$  and  $q$  is the perimeter of the triangle  $O_1O_2O_3$ . Prove that  $q \leq 6\sqrt{3}R$ . When does equality hold?

5. Let  $A_1, A_2, \dots, A_{2n}$  are the vertices of a regular  $2n$ -gon and  $P$  is a point from the incircle of the polygon. If  $\alpha_i = \angle A_i P A_{i+n}$ ,  $i = 1, 2, \dots, n$ . Prove the equality

$$\sum_{i=1}^n \tan^2 \alpha_i = 2n \frac{\cos^2 \frac{\pi}{2n}}{\sin^4 \frac{\pi}{2n}}$$

6. In a triangle pyramid  $SABC$  one of the plane angles with vertex  $S$  is a right angle and orthogonal projection of  $S$  on the base plane  $ABC$  coincides with orthocentre of the triangle  $ABC$ . Let  $SA = m$ ,  $SB = n$ ,  $SC = p$ ,  $r$  is the radii of incircle of  $ABC$ .  $H$  is the height of the pyramid and  $r_1, r_2, r_3$  are radii of the incircles of the intersections of the pyramid with the plane passing through  $SA, SB, SC$  and the height of the pyramid. Prove that

- (a)  $m^2 + n^2 + p^2 \geq 18r^2$  ;  
 (b)  $\frac{r_1}{H}, \frac{r_2}{H}, \frac{r_3}{H}$  are in the range  $(0.4, 0.5)$ .

**Note.** The last problem is proposed from Bulgaria for IMO and may be found at IMO Compendium book.

## Bulgarian Mathematical Olympiad 1972, III Round

### *First Day*

1. Prove that for every integer number and natural number  $n$  the number  $a^N - a$  is divisible by 13 where  $N = 2^{2^n} - 3$ . (6 points, Hr. Le Lesov)
2. Prove inequality:

$$1 + \frac{1}{1!\sqrt{2!}} + \frac{1}{2\sqrt{2!}\sqrt[3]{3!}} + \cdots + \frac{1}{(n-1)^{n-1}\sqrt{(n-1)!}\sqrt[n]{n}} > \frac{2(n^2 + n - 1)}{n(n+1)}$$

where  $n$  is a natural number, greater than 1. (7 points, Hr. Lesov)

3. Find all positive integer values of  $n$  for which whole plane may be covered with network that consists of regular  $n$ -gons. (7 points, Hr. Lesov)

### *Second day*

4. The opposite sides  $AB$  and  $CD$  of inscribed in the circle  $k$  quadrilateral  $ABCD$  intersects at a point  $M$ . Tangent  $MN$  ( $N$  belongs to  $k$ ) is parallel to the diagonal  $AC$ .  $NB$  intersects  $AC$  at the point  $P$ . Prove that the lines  $AN$ ,  $DB$  and  $PM$  intersects at a common point. (6 points, Hr. Lesov)
5. On the sides  $BC$ ,  $CA$ ,  $AB$  of acute angled triangle  $ABC$  externally are constructed squares which centers are denoted by  $M$ ,  $N$ ,  $P$ . Prove the inequality:

$$[MNP] \geq \left(1 + \frac{\sqrt{3}}{2}\right) [ABC]$$

(6 points, Hr. Lesov)

6. It is given a pyramid with base  $n$ -gon, circumscribed around a circle with center  $O$ , which is orthogonal projection of the vertex of the pyramid to the plane of the base of the pyramid. Prove that the orthogonal projections of  $O$  to the walls of the pyramid lies on the common circle.

(8 points, Hr. Lesov)

## Bulgarian Mathematical Olympiad 1972, IV Round

### *First Day*

1. Prove that there are no exists integer numbers  $a, b, c$  such that for every integer number  $x$  the number:  $A = (x + a)(x + b)(x + c) - x^3 - 1$  is divisible by 9. (Iv. Tonov)
2. Solve the system of equations:

$$\begin{cases} \sqrt{\frac{y(t-y)}{t-x} - \frac{4}{x}} + \sqrt{\frac{z(t-z)}{t-x} - \frac{4}{x}} = \sqrt{x} \\ \sqrt{\frac{z(t-z)}{t-y} - \frac{4}{y}} + \sqrt{\frac{x(t-x)}{t-y} - \frac{4}{y}} = \sqrt{y} \\ \sqrt{\frac{x(t-x)}{t-z} - \frac{4}{z}} + \sqrt{\frac{y(t-y)}{t-z} - \frac{4}{z}} = \sqrt{z} \\ x + y + z = 2t \end{cases}$$

if the following conditions are satisfied:  $0 < x < t, 0 < y < t, 0 < z < t$ .

(Hr. Lesov)

3. Prove the equality:

$$\sum_{k=1}^{n-1} \frac{1}{\sin^2 \frac{(2k+1)\pi}{2n}} = n^2$$

where  $n$  is a natural number.

(Hr. Lesov)

### *Second day*

4. Find maximal possible count of points which lying in or over a circle with radii  $R$  in such a way that the distance between every two points is greater than:  $R\sqrt{2}$ . (Hr. Lesov)
5. In a circle with radii  $R$  is inscribed a quadrilateral with perpendicular diagonals. From the intersecting point of the diagonals are drawn perpendiculars to the sides of the quadrilateral.
- (a) prove that the feet of these perpendiculars  $P_1, P_2, P_3, P_4$  are vertices of the quadrilateral that is inscribed and circumscribed.
- (b) Prove the inequalities:  $2r_1 \leq \sqrt{2}R_1 \leq R$  where  $R_1$  and  $r_1$  are radii respectively of the circumcircle and incircle to the quadrilateral:  $P_1P_2P_3P_4$ . When does equalities holds?
- (Hr. Lesov)
6. It is given a tetrahedron  $ABCD$  for which two points of opposite edges are mutually perpendicular. Prove that:
- (a) the four altitudes of  $ABCD$  intersects at a common point  $H$ ;
- (b)  $AH + BH + CH + DH < p + 2R$ , where  $p$  is the sum of the lengths of all edges of  $ABCD$  and  $R$  is the radii of circumscribed around  $ABCD$  sphere.

(Hr. Lesov)

# Bulgarian Mathematical Olympiad 1973, III Round

## *First Day*

1. In a library there are 20000 books ordered on the shelves in such a way that on each of the shelves there is at least 1 and at most 199 books. Prove that there exists two shelves with same count of books of them.

(L. Davidov)

2. Find the greatest common divisor of the numbers:

$$2^{2^2} + 2^{2^1} + 1, 2^{2^3} + 2^{2^2} + 1, \dots, 2^{2^{n+1}} + 2^{2^n} + 1, \dots$$

(Hr. Lesov)

3. Find all finite sets  $M$  of whole numbers that have at least one element and have the property: for every element  $x \in M$  there exists element  $y \in M$  for which the following equality is satisfied:  $4x^2 + 3 \leq 8y$ .

(Iv. Prodanov)

## *Second day*

4. Prove that if  $n$  is a random natural number and  $\alpha$  is number satisfying the condition:  $0 < \alpha < \frac{\pi}{n}$ , then:

$$\sin \alpha \sin 2\alpha \cdots \sin n\alpha < \frac{1}{n^n} \frac{1}{\sin^n \frac{\alpha}{2}}$$

(L. Davidov)

5. Through the center of gravity of the triangle  $ABC$  is drawn a line intersecting the sides  $BC$  and  $AC$  in the points  $M$  and  $N$  respectively. Prove that:

$$[AMN] + [BMN] \geq \frac{4}{9} [ABC]$$

When does equality holds?

(Hr. Lesov)



6. In a sphere with radii  $R$  is inscribed a regular  $n$ -angled pyramid. The angle between two adjacent (neighboring) edges is equal to:  $\frac{180^\circ}{n}$ . Express the ratio between the volume and the surface of the pyramid as a function to  $R$  and  $n$ .

## Bulgarian Mathematical Olympiad 1973, IV Round

### *First Day*

1. Let the sequence  $a_1, a_2, \dots, a_n, \dots$  is defined by the conditions:  $a_1 = 2$  and  $a_{n+1} = a_n^2 - a_n + 1$  ( $n = 1, 2, \dots$ ). Prove that:
- (a)  $a_m$  and  $a_n$  are relatively prime numbers when  $m \neq n$ .
- (b)  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{a_k} = 1$ .

(Iv. Tonov)

2. Let the numbers  $a_1, a_2, a_3, a_4$  form an arithmetic progression with difference  $d \neq 0$ . Prove that there are no exists geometric progressions  $b_1, b_2, b_3, b_4$  and  $c_1, c_2, c_3, c_4$  such that:

$$a_1 = b_1 + c_1, a_2 = b_2 + c_2, a_3 = b_3 + c_3, a_4 = b_4 + c_4$$

3. Let  $a_1, a_2, \dots, a_n$  are different integer numbers in the range:  $[100, 200]$  for which:  $a_1 + a_2 + \dots + a_n \geq 11100$ . Prove that it can be found at least number from the given in the representation of decimal system on which there are at least two equal (same) digits. (L. Davidov)

### *Second day*

4. Find all functions  $f(x)$  defined in the range  $(-\frac{\pi}{2}, \frac{\pi}{2})$  they can be differentiated for  $x = 0$  and satisfy the condition:

$$f(x) = \frac{1}{2} \left( 1 + \frac{1}{\cos x} \right) f\left(\frac{x}{2}\right)$$

for every  $x$  in the range  $(-\frac{\pi}{2}, \frac{\pi}{2})$ . (L. Davidov)

5. Let the line  $\ell$  intersects the sides  $AC$ ,  $BC$  of the triangle  $ABC$  respectively at the points  $E$  and  $F$ . Prove that the line  $\ell$  is passing through the incenter of the triangle  $ABC$  if and only if the following equality is true:

$$BC \cdot \frac{AE}{CE} + AC \cdot \frac{BF}{CF} = AB$$

(H. Lesov)

6. In the tetrahedron  $ABCD$ ,  $E$  and  $F$  are the middles of  $BC$  and  $AD$ ,  $G$  is the middle of the segment  $EF$ . Construct a plane through  $G$  intersecting the segments  $AB$ ,  $AC$ ,  $AD$  in the points  $M$ ,  $N$ ,  $P$  respectively in such a way that the sum of the volumes of the tetrahedrons  $BMNP$ ,  $CMNP$  and  $DMNP$  to be a minimal.  
(Hr. Lesov)

## Bulgarian Mathematical Olympiad 1974, III Round

### *First Day*

1. The sequence  $\{a_n\}$  is defined in the following way:  $a_0 = 2$  and

$$a_{n+1} = a_n + \frac{1}{a_n}, \quad n = 0, 1, 2, \dots$$

Prove that  $62 < a_{1974} < 77$ . (5 Points, I. Tonov)

2. It's given the triangle  $ABC$ . In its sides externally are constructed similar triangles  $ABK$ ,  $BCL$ ,  $CAM$  (it is know that  $AB : BC : CA = KA : LB : MC$ ). Prove that the centers of gravity of the triangles  $ABC$  and  $KLM$  coincides. (7 Points, L. Davidov)
3. Let  $n$  and  $k$  are natural numbers such that  $k \geq 2$ . Prove that there exists  $n$  sequential natural numbers, such that every one of them may be presented as a multiple of at least  $k$  prime multipliers. (8 Points, V. Chukanov)

### *Second day*

4. Find the natural number  $x$  defined by the equality:

$$\left[ \sqrt[3]{1} \right] + \left[ \sqrt[3]{2} \right] + \dots + \left[ \sqrt[3]{x^3 - 1} \right] = 400$$

(6 Points, V. Petnov)

5. In a cube with edge 9 are *thrown* 40 regular phrisms with side of the base 1,5 and a height not greater than 1,4. Prove that there exists a sphere with radius 0,5 lying in the cube and not having common points with the prisms. (6 Points, H. Lesov)
6. In a plane are given circle  $k$  with centre  $O$  and point  $P$  lying outside  $k$ . There are constructed tangents  $PQ$  and  $PR$  from  $P$  to  $k$  and on the smaller arc  $QR$  is chosen a random point ( $B$ ,

$B \neq Q, B \neq R$ ). Through point  $B$  is constructed a tangent to  $k$  intersecting  $PQ$  and  $PR$  respectively at points  $A$  and  $C$ . Prove that the length of the segment  $QR$  is equal to the minimal perimeter of inscribed  $AOC$  triangles.

(8 points, L. Davidov)

## Bulgarian Mathematical Olympiad 1974, IV Round

### *First Day*

1. Find all natural numbers  $n$  with the following property: there exists a permutation  $(i_1, i_2, \dots, i_n)$  of the numbers  $1, 2, \dots, n$  such that, if on the circular table sit  $n$  people and for all  $k = 1, 2, \dots, n$  the  $k$ -th person is moving in places in right, all people will sit on different places. (V. Drenski)

2. Let  $f(x)$  and  $g(x)$  are non constant polynomials with integer positive coefficients,  $m$  and  $n$  are given natural numbers. Prove that there exists infinitely many natural numbers  $n$  for which the numbers the numbers

$$f(m^n) + g(0), f(m^n) + g(1), \dots, f(m^n) + g(k)$$

are composite. (I. Tonov)

3. (a) Find all real numbers  $p$  for which the inequality

$$x_1^2 + x_2^2 + x_3^2 \geq p(x_1x_2 + x_2x_3)$$

is true for all real numbers  $x_1, x_2, x_3$ .

- (b) Find all real numbers  $q$  for which the inequality

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 \geq q(x_1x_2 + x_2x_3 + x_3x_4)$$

is true for all real numbers  $x_1, x_2, x_3, x_4$ .

(I. Tonov)

### *Second day*

4. Find the maximal count of shapes that can be placed over a chessboard with size  $8 \times 8$  in such a way that no three shapes are not on two squares, lying next to each other by diagonal parallel  $A1 - H8$  ( $A1$  is the lowest-bottom left corner of the chessboard,  $H8$  is the highest-upper right corner of the chessboard). (V. Chukanov)
5. Find all point  $M$  lying into given acute-angled triangle  $ABC$  and such that the surface of the triangle with vertices on the foots of the perpendiculars drawn from  $M$  to the lines  $BC$ ,  $CA$ ,  $AB$  is maximal. (H. Lesov)
6. In triangle pyramid  $MABC$  at least two of the plane angles next to the edge  $M$  are not equal to each other. Prove that if the angles bisectors of these angles form the same angle with the angle bisector of the third plane angle, the following inequality is true

$$8a_1b_1c_1 \leq a^2a_1 + b^2b_1 + c^2c_1$$

where  $a$ ,  $b$ ,  $c$  are sides of triangle  $ABC$  and  $a_1, b_1, c_1$  are edges crossed respectively with  $a$ ,  $b$ ,  $c$ . (V. Petkov)

## Bulgarian Mathematical Olympiad 1975, III Round

### *First Day*

1. Let  $n$  is an odd natural number and  $a_1, a_2, \dots, a_n$  is a permutation of the numbers  $1, 2, \dots, n$ . Prove that the number

$$(a_1 - 1)(a_2 - 3) \cdots (a_n - n)$$

is an even number. (L. Davidov)

2. Let  $m, n, p$  are three sides of a billiard table with the shape of an equilateral triangle. A ball is situated at the middle of  $m$  side and is directed to the side  $n$  under angle  $\alpha$  (the angle of the trajectory of the ball and  $n$  is  $\alpha$ ). For which values of  $\alpha$  the ball after its reflection on  $n$  will reach the side  $p$  and after its reflection will reach the side  $m$ ? (V. Petkof)

3. Prove that the number  $2^{147} - 1$  is divisible by 343. (V. Chukanov)

### *Second day*

4. Find all polynomials  $f(x)$ , satisfying the conditions  $f(2x) = f'(x)f''(x)$ . (L. Davidov)

5. Calculate:

$$\left[ \frac{1}{2} \left( 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \cdots + \frac{1}{\sqrt{10000}} \right) \right]$$

where  $[\alpha]$  is the integer part of the number  $\alpha$  (biggest integer number not bigger than  $\alpha$ ). A possible way to calculate that number is to prove and use the inequality

$$2 \left( \sqrt{n+1} - \sqrt{n} \right) < \frac{1}{\sqrt{n}} < 2 \left( \sqrt{n} - \sqrt{n-1} \right)$$

for all integer numbers  $n$ . (I. Prodanov)

6. In a regular  $n$ -angled truncated pyramid we may inscribe a sphere touching all walls and can be found other sphere touching all edges of pyramid

- (a) Prove that  $n = 3$ ;
- (b) find dihedral angle between a surrounding wall and the biggerbase.

(V. Petkov)

## Bulgarian Mathematical Olympiad 1975, IV Round

### *First Day*

1. Find all pairs of natural numbers  $(m, n)$  bigger than 1 for which  $2^m + 3^n$  is a square of whole number. (I. Tonov)
2. Let  $F$  is polygon the boundary of which is a broken line with vertices in the knots (units) of a given in advance regular square network. If  $k$  is the count of knots of the network situated over the boundary of  $F$ , and  $l$  is the count of the knots of the network lying inside  $F$ , prove that if the surface of every square from the network is 1, then the surface  $S$  of  $F$  is calculated with the formulae:

$$S = \frac{k}{2} + l - 1$$

(V. Chukanov)

3. Let  $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3$  is a polynomial with real coefficients ( $a_0 \neq 0$ ) and such that  $|f(x)| \leq 1$  for every  $x \in [-1, 1]$ . Prove that
  - (a) there exist a constant  $c$  (one and the same for all polynomials with the given property), for which  $|a_i| \leq c$ ,  $i = 0, 1, \dots$
  - (b)  $|a_0| \leq 4$ .

(V. Petkov)

### *Second day*

4. In the plane are given a circle  $k$  with radii  $R$  and the points  $A_1, A_2, \dots, A_n$ , lying on  $k$  or outside  $k$ . Prove that there exist infinitely many points  $X$  from the given circumference for which

$$\sum_{i=1}^n A_i X^2 \geq 2nR^2$$

Is there exist a pair of points on different sides of some diameter,  $X$  and  $Y$  from  $k$ , such that

$$\sum_{i=1}^n A_i X^2 \geq 2nR^2 \text{ and } \sum_{i=1}^n A_i Y^2 \geq 2nR^2?$$

(H. Lesov)

5. Let *subbishop* (bishop is the figure moving only by a diagonal) is a figure moving only by diagonal but only in the next cells (sugres) of the chessboard. Find the maximal count of subbishops over a chessboard  $n \times n$ , no two of which are not attacking. (V. Chukanov)
6. Some of the walls of a convex polyhedron  $M$  are painted in blue, others are painted in white and there are no two walls with common edge. Prove that if the sum of surfaces of the blue walls is bigger than half surface of  $M$  then it may be inscribed a sphere in the polyhedron given ( $M$ ). (H. Lesov)



# Bulgarian Mathematical Olympiad 1976, III Round

## *First Day*

1. Let  $\alpha$  is a positive number. Prove that

$$\sqrt[3]{27 + 8\alpha} < \sqrt[3]{1 + \alpha} + \sqrt[3]{8 + \alpha}$$

(8 points, J. Tabov)

2. In a triangle pyramid  $MABC$ :  $\angle AMB = \angle BMC = \angle CMA = 90^\circ$ . If  $h$  is the height of the pyramid from  $M$  to  $ABC$ ,  $r$  is the radii of inscribed in the pyramid sphere and  $V$  is its volume. Prove that

$$V \geq \frac{9h^3 r^3}{2(h - r)^3}$$

where equality occurs only if the three edges passing through  $M$  are all equal to each other. (7 points)

3. Let natural numbers  $m$  and  $n$  are sidelengths of the rectangle  $P$  divided to  $mn$  squares with side 1 with lines parallel to its sides. Prove that the diagonal of the rectangle passes through internal points of  $m + n - d$  squares, when  $d$  is the biggest common divisor of  $m$  and  $n$ .

(6 points, I. Tonov)

## *Second day*

4. It is given the system

$$\begin{cases} x^2 - |x|\sqrt{y} - y\sqrt{y} = 0 \\ x^2 - x(y + 2y\sqrt{y}) + y^3 = 0 \end{cases}$$

Find all real solutions  $(x, y)$  of the system for which  $x \neq 1$ . (8 points)

5. Let the point  $P$  is internal for the circle  $k$  with center  $O$ . The cord  $AB$  is passing through  $P$ . Prove that  $\tan \frac{\alpha}{2} \tan \frac{\beta}{2}$ , where  $\alpha = \angle AOP$ ,  $\beta = \angle BOP$  is a constant for all chords passing through  $P$ . (4 points)
6. In an acute angled triangle through its orthocentre is drawn a line  $\ell$  which intersects two sides of the triangle and the continuation of third side. There are drawn the lines symmetric of  $\ell$  about the sides of the triangle. Prove that these lines intersects at a common point from circumscribed circle of the triangle  $ABC$ . (7 points, V. Petkov)

## Bulgarian Mathematical Olympiad 1976, IV Round

### *First Day*

1. In a circle with radius 1 is inscribed hexagon (convex). Prove that if the multiple of all diagonals that connects vertices of neighbour sides is equal to 27 then all angles of hexagon are equals. (5 points, V. Petkov, I. Tonov)
2. Find all polynomials  $p(x)$  satisfying the condition:

$$p(x^2 - 2x) = (p(x - 2))^2$$

(7 points)

3. In the space is given a tetrahedron with length of the edge 2. Prove that distances from a random point  $M$  to all of the vertices of the tetrahedron are integer numbers if and only if  $M$  is a vertex of tetrahedron. (8 points, J. Tabov)

### *Second day*

4. Let  $0 < x_1 \leq x_2 \leq \dots \leq x_n$ . Prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \dots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \geq \frac{x_2}{x_1} + \frac{x_3}{x_2} + \dots + \frac{x_n}{x_{n-1}} + \frac{x_1}{x_n}$$

(7 points, I. Tonov)

5. It is given a tetrahedron  $ABCD$  and a plane  $\alpha$  intersecting the three edges passing through  $D$ . Prove that  $\alpha$  divides surface of tetrahedron to two parts proportional to the volumes of the bodies formed if and only if  $\alpha$  is passing through the center of the inscribed tetrahedron sphere.

(6 points, H. Lesov)

6. It is given a plane with coordinate system with a beginning at the point  $O$ .  $A(n)$ , when  $n$  is a natural number is a count of the points with whole coordinates which distances to  $O$  is less than or equal to  $n$ .

- (a) Find

$$\ell = \lim_{n \rightarrow \infty} \frac{A(n)}{n^2}$$

- (b) For which  $\beta$  ( $1 < \beta < 2$ ) there exists the limit

$$\lim_{n \rightarrow \infty} \frac{A(n) - \pi n^2}{n^\beta}$$

(7 points)

# Bulgarian Mathematical Olympiad 1977, III Round

## *First Day*

1. It is given a pyramid with a base quadrilateral. In each of the walls of the pyramid is inscribed a circle. It is also known that inscribed circles of each two of the walls with common edge have a common point on that edge. Prove that four points in that inscribed in the four walls circles are tangent to the base lies on a common circle. (Here walls are the four side walls of the pyramid and the base of the pyramid is not a wall).  
(7 points)
2. Find all integer numbers  $x$  for which  $x^2 + 1$  divides  $x^3 - 8x^2 + 2x$  without remainder.  
(6 points)
3. Prove that when the number  $(7 + 4\sqrt{3})^n$ ,  $n \geq 1$ ,  $n \in \mathbb{N}$  is written in decimal system the digit 9 occurs in it at least  $n$  times after the decimal point.  
(7 points)

## *Second day*

4. Find all real solutions of the system:

$$\begin{cases} \frac{2x^2}{1+x^2} = y \\ \frac{2y^2}{1+y^2} = z \\ \frac{2z^2}{1+z^2} = x \end{cases}$$

(7 points)

5. There are given two circumferences  $k_1$  and  $k_2$  with centers  $O_1$  and  $O_2$  respectively with different radii that are tangent outside each other at a point  $A$ . It is given a point  $M$  inside  $k_1$ , not lying on the line  $O_1O_2$ . Construct a line  $\ell$  that passes through  $M$  and for which circumscribed circle with vertices  $A$  and two of the common points of  $\ell$  with  $k_1$  and  $\ell$  with  $k_2$  is tangent to the line  $O_1O_2$ .  
(7 points, Jordan Tabov)

6. In a group of people two persons  $X, Y$  are named *non-directly known* if they do not know each other (themselves) personally or if there exists a chain of people  $Z_1, Z_2, \dots, Z_p$  such that  $X$  and  $Z_1$  are known,  $Z_1$  and  $Z_2$  are known,  $\dots$ ,  $Z_p$  and  $Y$  are known. Let the group consist of 134 persons and for each 8 of them at least two are *non-directly known*. Prove that there exists a group of 20 people every two of which are *non-directly known*.

(6 points, N. Nenov, N. Hadzhiivanov)

## Bulgarian Mathematical Olympiad 1977, IV Round

### First Day

1. Let for natural number  $n$  and real numbers  $\alpha$  and  $x$  be satisfied inequalities  $\alpha^{n+1} \leq x \leq 1$  and  $0 < \alpha < 1$ . Prove that

$$\prod_{k=1}^n \left| \frac{x - \alpha^k}{x + \alpha^k} \right| \leq \prod_{k=1}^n \left| \frac{1 - \alpha^k}{1 + \alpha^k} \right|$$

(7 points, Borislav Boyanov)

2. In the space are given  $n$  points and no four of them belongs to a common plane. Some of the points are connected with segments. It is known that four of the given points are vertices of tetrahedron whose edges belong to the segments given. It is also known that common number of the segments, passing through vertices of tetrahedron is  $2n$ . Prove that there exists at least two tetrahedrons every one of which has a common wall with the first (initial) tetrahedron.

(6 points, N. Nenov, N. Hadzhiivanov)

3. It is given truncated pyramid whose bases are triangles with the base a triangle. The areas of the bases are  $B_1$  and  $B_2$  and the area of rounded surface is  $S$ . Prove that if there exists a plane parallel to the bases which intersection divides the pyramid into two truncated pyramids in which may be inscribed spheres then

$$S = \left( \sqrt{B_1} + \sqrt{B_2} \right) \left( \sqrt[4]{B_1} + \sqrt[4]{B_2} \right)^2$$

(7 points, G. Gantchev)

*Second day*

4. Vertices  $A$  and  $C$  of the quadrilateral  $ABCD$  are fixed points of the circle  $k$  and each of the vertices  $B$  and  $D$  is moving to one of the arcs of  $k$  with ends  $A$  and  $C$  in such a way that  $BC = CD$ . Let  $M$  is the intersection point of  $AC$  and  $BD$  and  $F$  is the centre of circumscribed circle around  $\triangle ABM$ . Prove that the locus of  $F$  is an arc of a circle.  
(7 points, J. Tabov)
5. Let  $Q(x)$  is a non-zero polynomial and  $k$  is a natural number. Prove that the polynomial  $P(x) = (x - 1)^k Q(x)$  have at least  $k + 1$  non-zero coefficients.  
(7 points)
6. Pythagor's triangle is every right-angled triangle for which the lengths of two cathets and the length of the hypotenuse are integer numbers. We are observing all Pythagor's triangles in which may be inscribed a quadrangle with sidelength integer number, two of which sides lies on the cathets and one of the vertices of which lies on the hypotenuse of the triangle given. Find the sidelengths of the triangle with minimal surface from the observed triangles.  
(6 points, St. Dodunekov)

# Bulgarian Mathematical Olympiad 1978, III Round

## *First Day*

1. There are given 10000 natural numbers forming an arithmetic progression which common difference is an odd number not divisible by 5. Prove that one of the given numbers ends with 1978 (1978 are last four digits in decimal counting system). (6 points, S. Dodunekov)
2. Circles  $c_1$  and  $c_2$  which radiuses are  $r_1$  and  $r_2$  respectively ( $r_1 > r_2$ ) are tangent to each other internally. A line intersects circumferences in points  $A, B, C, D$  (the points are in this order on the line). Find the length of the segment  $AB$  if  $AB : BC : CD = 1 : 2 : 3$  and the centers of the circumferences are on one and the same side of the line. (7 points, G. Ganchev)
3. In the space are given  $n$  points in common position (there are no 4 points of them that are on one and the same plane). We observe all possible tetrahedrons all 4 vertices of which are from given points. Prove that if a plane contains no point from the given  $n$  points then the plane intersects at most

$$\frac{n^2(n-2)}{64}$$

from these tetrahedrons in quadrilaterals.

(7 points, N. Nenov, N. Hadzhiivanov)

## *Second day*

4. Find all possible real values of  $p, q$  for which the solutions of the equation:

$$x^3 - px^2 + 11x - q = 0$$

are three consecutive whole numbers. (6 points, Jordan Tabov)

5. It is given a pyramid which base is a rhombus  $OABC$  with side length  $a$ . The edge is perpendicular to the base plane (containing  $OABC$ ). From  $O$  are drawn perpendiculars  $OP$ ,  $OQ$ ,  $OS$  respectively to edges  $MA$ ,  $MB$ ,  $MC$ . Find the lengths of the diagonals of  $OABC$  if it is known that  $OP$ ,  $OQ$ ,  $OS$  lies in one and the same plane. (7 points, G. Gantchev)
6. Prove that the number  $\cos \frac{5\pi}{7}$  is:
- (a) root of the equation:  $8x^3 - 4x^2 - 4x + 1 = 0$ ;
- (b) irrational number.

(7 points)

## Bulgarian Mathematical Olympiad 1978, IV Round

### *First Day*

1. It is given the sequence:  $a_1, a_2, a_3, \dots$ , for which:  $a_n = \frac{a_{n-1}^2 + c}{a_{n-2}}$  for all  $n > 2$ . Prove that the numbers  $a_1$ ,  $a_2$  and  $\frac{a_1^2 + a_2^2 + c}{a_1 a_2}$  are whole numbers. (6 points)
2.  $k_1$  denotes one of the arcs formed by intersection of the circumference  $k$  and the chord  $AB$ .  $C$  is the middle point of  $k_1$ . On the half line (ray)  $PC$  is drawn the segment  $PM$ . Find the locus formed from the point  $M$  when  $P$  is moving on  $k_1$ . (7 points, G. Ganchev)
3. On the name day of a men there are 5 people. The men observed that of any 3 people there are 2 that knows each other. Prove that the man may order his guests around circular table in such way that every man have on its both side people that he knows. (6 points, N. Nenov, N. Hazhiivanov)

### *Second day*



4. Find greatest possible real value of  $S$  and smallest possible value of  $T$  such that for every triangle with sides  $a, b, c$  ( $a \leq b \leq c$ ) to be true the inequalities:

$$S \leq \frac{(a + b + c)^2}{bc} \leq T$$

(7 points)

5. Prove that for every convex polygon can be found such a three sequential vertices for which a circle that they lies on covers the polygon.

(7 points, Jordan Tabov)

6. The base of the pyramid with vertex  $S$  is a pentagon  $ABCDE$  for which  $BC > DE$  and  $AB > CD$ . If  $AS$  is the longest edge of the pyramid prove that  $BS > CS$ . (7 points, Jordan Tabov)

## Bulgarian Mathematical Olympiad 1991, III Round

### First Day

1. Prove that if  $x_1, x_2, \dots, x_k$  are mutually different (there are no two equal) (it is not required to be sequential) members of the arithmetic progression  $2, 5, 8, 11, \dots$  for which:

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_k} = 1$$

then its number is greater than 8 ( $k \geq 8$ ).

2. On the hypotenuse  $AB$  of a right-angled triangle  $ABC$  is fixed a point  $P$ . Let  $\ell^{\rightarrow}$  is a ray formed by the line  $BC$  with starting point  $C$  and not containing the point  $B$ . For each point  $T \neq C$  from  $\ell^{\rightarrow}$  with  $S$  is denoted the intersection point of the lines  $PT$  and  $AC$  and  $M$  is the intersection point of the lines  $BS$  and  $AT$ . Find the locus of the point  $M$  when  $T$  is moving on the ray  $\ell^{\rightarrow}$ .

3. Let  $Oxy$  is a right-angled plane coordinate system. A point  $A(x, y)$  is called rational if its coordinates are rational numbers (for example the point  $A_0(-1, 0)$  is a rational point). Let  $k$  is a circle with the beginning of the coordinate system and with radii 1.

- (a) Prove that  $A(x, y) \neq A_0$  is a rational point from  $k$  if and only if

$$x = \frac{1 - p^2}{1 + p^2}, \quad y = \frac{2p}{1 + p^2}$$

for some rational number  $p$ .

- (b) Find an infinite sequence  $A_1(x_1, y_1), A_2(x_2, y_2), \dots, A_n(x_n, y_n), \dots$  that consists from mutually different points from  $k$  in such a way that  $\lim_{n \rightarrow \infty} A_n = A_0$  (i. e.  $\lim_{n \rightarrow \infty} x_n = -1$  and  $\lim_{n \rightarrow \infty} y_n = 0$  and the length of the segment  $A_n A_0$  is a rational number for:  $n = 1, 2, \dots$ ).
- (c) Prove that for each arc from  $k$  we may choose at least two rational points such that the length of the distance between them is a rational number.

*Second day*

4. (a) Prove that if  $a, b, c$  are positive real numbers for which the following inequality is satisfied

$$(a^2 + b^2 + c^2)^2 > 2(a^4 + b^4 + c^4)$$

then there exists a triangle with sides  $a, b$  and  $c$ .

- (b) Prove that if  $a, b, c, d$  are positive real numbers for which the following inequality is satisfied

$$(a^2 + b^2 + c^2 + d^2)^2 > 3(a^4 + b^4 + c^4 + d^4)$$

then can be formed a triangle with sides equal to any three from the numbers given.

5. Prove that if from the angle bisectors of a given triangle can be constructed a triangle similar to the triangle given. Then the triangle given is an equilateral triangle.
6. Let the natural number  $n \geq 3$  is presented as a sum of  $k \geq 2$  natural numbers

$$n = x_1 + x_2 + \cdots + x_k$$

in such a way that  $x_i \leq \frac{n}{2}$  for each  $i = 1, 2, \dots, k$ . Prove that the vertices of an  $n$ -gon can be colored in  $k$ -colors in such a way that  $x_1$  vertices are colored in the first color,  $x_2$  vertices are colored in the second color,  $\dots, x_k$  vertices are colored in the  $k$ -th color and every two different vertices are colored in different colors.

## Bulgarian Mathematical Olympiad 1992, III Round

*First Day*

1. Prove that the numbers

$$10101, \quad 1010101, \quad 10101 \cdots 101, \quad \dots$$

are composite ones.

2. Let  $a$  and  $b$  positive numbers. Prove that the inequality  $\sqrt{a}+1 > \sqrt{b}$  holds if and only if for every  $x > 1$  the inequality

$$ax + \frac{x}{x-1} > b$$

holds.

3. The hexagon  $ABCDEF$  is inscribed in a circle so that  $|AB| = |CD| = |EF|$ . Let  $P, Q, R$  be points of intersection of the diagonals  $AC$  and  $BD, CE$  and  $DF, EA$  and  $FB$  respectively. Prove that the triangles  $PQR$  and  $BDF$  are similar,

*Second day*

4. Prove that the sum of squares 3,4,5 or 6 consecutive integers is not a perfect square. Give an example of 11 consecutive integers such that the sum of their squares is a perfect square.
5. A convex 15-gon is given. Prove that at least two of its diagonals lie on lines forming an angle not greater than  $2^\circ$ .
6. Let us denote by  $S(x)$  the sum of the digits of the positive integer  $x$  in a decimal positional system.

- (a) Prove that for every positive integer  $x$  the following inequality hold,

$$\frac{S(x)}{S(2x)} \leq 5$$

Is this estimation accurate ?

- (b) Prove that the function  $\frac{S(x)}{S(3x)}$  is unbounded.

## Bulgarian Mathematical Olympiad 1992, IV Round

### *First Day*

1. Through a random point  $C_1$  from the edge  $DC$  of the regular tetrahedron  $ABCD$  is drawn a plane, parallel to the plane  $ABC$ . The plane constructed intersects the edges  $DA$  and  $DB$  at the points  $A_1$ ,  $B_1$  respectively. Let the point  $H$  is the midpoint of the height through the vertex  $D$  of the tetrahedron  $DA_1B_1C_1$  and  $M$  is the center of gravity (medicenter) of the triangle  $ABC_1$ . Prove that the dimension of the angle  $HMC$  doesn't depend of the position of the point  $C_1$ .  
(Ivan Tonov)
2. Prove that there exists 1904-element subset of the set  $\{1, 2, \dots, 1992\}$ , which doesn't contain an arithmetic progression consisting of 41 terms.  
(Ivan Tonov)
3. Let  $m$  and  $n$  are fixed natural numbers and  $Oxy$  is a coordinate system in the plane. Find the total count of all possible situations of  $n+m-1$  points  $P_1(x_1, y_1), P_2(x_2, y_2), \dots, P_{n+m-1}(x_{n+m-1}, y_{n+m-1})$  in the plane for which the following conditions are satisfied:
  - (i) The numbers  $x_i$  and  $y_i$  ( $i = 1, 2, \dots, n + m - 1$ ) are integer (whole numbers) and  $1 \leq x_i \leq n, 1 \leq y_i \leq m$ .
  - (ii) Every one of the numbers  $1, 2, \dots, n$  can be found in the sequence  $x_1, x_2, \dots, x_{n+m-1}$  and every one of the numbers  $1, 2, \dots, m$  can be found in the sequence  $y_1, y_2, \dots, y_{n+m-1}$ .
  - (iii) For every  $i = 1, 2, \dots, n + m - 2$  the line  $P_iP_{i+1}$  is parallel to one of the coordinate axes.

(Ivan Gochev, Hristo Minchev)

### *Second day*

4. Let  $p$  is a prime number in the form  $p = 4k + 3$ . Prove that if the numbers  $x_0, y_0, z_0, t_0$  are solution of the equation:  $x^{2p} + y^{2p} + z^{2p} = t^{2p}$ , then at least one of them is divisible by  $p$ . (Plamen Koshlukov)
5. Points  $D, E, F$  are middlepoints of the sides  $AB, BC, CA$  of the triangle  $ABC$ . Angle bisectors of the angles  $BDC$  and  $ADC$  intersects the lines  $BC$  and  $AC$  respectively at the points  $M$  and  $N$  and the line  $MN$  intersects the line  $CD$  at the point  $O$ . Let the lines  $EO$  and  $FO$  intersects respectively the lines  $AC$  and  $BC$  at the points  $P$  and  $Q$ . Prove that  $CD = PQ$ . (Plamen Koshlukov)
6. There are given one black box and  $n$  white boxes ( $n$  is a random natural number). White boxes are numbered with the numbers  $1, 2, \dots, n$ . In them are put  $n$  balls. It is allowed the following rearrangement of the balls: if in the box with number  $k$  there are exactly  $k$  balls that box is made empty - one of the balls is put in the black box and the other  $k - 1$  balls are put in the boxes with numbers:  $1, 2, \dots, k - 1$ . (Ivan Tonov)

# Bulgarian Mathematical Olympiad 1993, III Round

*First Day, 24 april 1993*

1. Prove that the equation:

$$x^3 - y^3 = xy + 1993$$

don't have a solution in positive integers.

2. It is given a right-angled triangle  $ABC$ .  $AC$  and  $BC$  are its cathetus.  $M$  is the midpoint of  $BC$ . A circle  $k$  passing through  $A$  and  $M$  is tangent to the circumcircle of  $ABC$ .  $N$  is the second point of intersection of  $k$  and the line  $BC$ . Prove that the line  $AN$  is passing through the midpoint of the height  $CH$  of the triangle  $ABC$ .
3. Prove that if  $a, b, c$  are positive numbers and  $p, q, r \in [0, 1]$  and  $a + b + c = p + q + r = 1$ , then

$$abc \leq \frac{pa + qb + rc}{8}.$$

*Second day, 25 april 1993*

4. Let  $a, b, c$  are positive numbers for which:  $9a + 11b + 29c = 0$ . Prove that the equation  $4ax^3 + bx + c = 0$  have a real root in the closed interval  $[0, 2]$ .
5. It is given an acute-angled triangle  $ABC$  for which  $BC = AC\sqrt{2}$ . Through the vertex  $C$  are drawn lines  $\ell$  and  $m$  (different from the lines  $AC$  and  $BC$ ), which intersects the line  $AB$  respectively at the points  $L$  and  $M$  in such a way that  $AL = MB$ . The lines  $\ell$  and  $m$  intersects circumcircle of  $ABC$  at the points  $P$  and  $Q$  respectively and the lines  $PQ$  and  $AB$  intersects each other at  $N$  prove that  $AN = NB$ .

6. It is given a convex hexagon with sidelength equal to 1. Find biggest natural number  $n$  for which internally to the hexagon given can be situated  $n$  points in such a way that the distance between any two of them is not less than  $\sqrt{2}$ .

## Bulgarian Mathematical Olympiad 1993, IV Round

### *First Day*

1. Find all functions  $f$ , defined and having values in the set of integer numbers, for which the following conditions are satisfied:
- (a)  $f(1) = 1$ ;
  - (b) for every two whole (integer) numbers  $m$  and  $n$ , the following equality is satisfied:

$$f(m+n) \cdot (f(m) - f(n)) = f(m-n) \cdot (f(m) + f(n))$$

2. The point  $M$  is internal point for the triangle  $ABC$  such that:  $\angle AMC = 90^\circ$ ,  $\angle AMB = 150^\circ$  and  $\angle BMC = 120^\circ$ . Points  $P$ ,  $Q$  and  $R$  are centers of circumscribed circles around triangles  $AMC$ ,  $AMB$  and  $BMC$ . Prove that the area of triangle  $PQR$  is bigger than the area of the triangle  $ABC$ .
3. It is given a polyhedral constructed from two regular pyramids with bases heptagons (a polygon with 7 vertices) with common base  $A_1A_2A_3A_4A_5A_6A_7$  and vertices respectively the points  $B$  and  $C$ . The edges  $BA_i$ ,  $CA_i$  ( $i = 1, \dots, 7$ ), diagonals of the common base are painted in blue or red. Prove that there exists three vertices of the polyhedral given which forms a triangle with all sides in the same color.

### *Second day*

4. Find all natural numbers  $n > 1$  for which there exists such natural numbers  $a_1, a_2, \dots, a_n$  for which the numbers  $\{a_i + a_j \mid 1 \leq i < j \leq n\}$  forms a full system modulo  $\frac{n(n+1)}{2}$ .



5. Let  $Oxy$  is a fixed rectangular coordinate system in the plane. Each ordered pair of points  $A_1, A_2$  from the same plane which are different from  $O$  and have coordinates  $x_1, y_1$  and  $x_2, y_2$  respectively is associated with real number  $f(A_1, A_2)$  in such a way that the following conditions are satisfied:
- (a) If  $OA_1 = OB_1, OA_2 = OB_2$  and  $A_1A_2 = B_1B_2$  then  $f(A_1, A_2) = f(B_1, B_2)$ .
  - (b) There exists a polynomial of second degree  $F(u, v, w, z)$  such that  $f(A_1, A_2) = F(x_1, y_1, x_2, y_2)$ .
  - (c) There exists such a number  $\phi \in (0, \pi)$  that for every two points  $A_1, A_2$  for which  $\angle A_1OA_2 = \phi$  is satisfied  $f(A_1, A_2) = 0$ .
  - (d) If the points  $A_1, A_2$  are such that the triangle  $OA_1A_2$  is equilateral with side 1 then  $f(A_1, A_2) = \frac{1}{2}$ .

Prove that  $f(A_1, A_2) = \overrightarrow{OA_1} \cdot \overrightarrow{OA_2}$  for each ordered pair of points  $A_1, A_2$ .

6. Find all natural numbers  $n$  for which there exists set  $S$  consisting of  $n$  points in the plane, satisfying the condition:
- For each point  $A \in S$  there exist at least three points say  $X, Y, Z$  from  $S$  such that the segments  $AX, AY$  and  $AZ$  have length 1 (it means that  $AX = AY = AZ = 1$ ).

## Bulgarian Mathematical Olympiad 1994, III Round

*First Day, 23 april 1994*

1. Let  $n > 1$  is a natural number and  $A_n = \{x \in \mathbb{N} \mid (x, n) \neq 1\}$  is the set of all natural numbers that aren't mutually prime (co-prime) with  $n$ . We say that the  $n$  is *interesting* if for every two numbers  $x, y \in A_n$  it is true  $x + y \in A_n$ .
  - (a) Prove that the number 43 is *interesting*.
  - (b) Prove that 1994 isn't *interesting*.
  - (c) Find all *interesting* numbers.
2. Around some circle are circumscribed a square and a triangle. Prove that at least a half from the square's perimeter lies inside the triangle.
3. Around unlimited chessboard is situated the figure  $(p, q)$ -horse which on each its move moves  $p$ -fields horizontally or vertically and  $q$ -fields on direction perpendicular to the previous direction (the ordinary chess horse is  $(2, 1)$ -horse). Find all pairs of natural numbers  $(p, q)$  for which the  $(p, q)$ -horse can reach to all possible fields on the chessboard with limited count of moves.

*Second day, 24 april 1994*

4. The sequence  $a_0, a_1, \dots, a_n$  satisfies the condition:

$$a_{n+1} = 2^n - 3a_n \quad , \quad n = 0, 1,$$

- (a) Express the common term  $a_n$  as a function of  $a_0$  and  $n$ .
  - (b) Find  $a_0$  if  $a_{n+1} > a_n$  for each natural number  $n$ .
5. It is given a rectangular parallelepiped  $ABCD A_1 B_1 C_1 D_1$ . The perpendiculars from the point  $A$  to the lines  $A_1 B$ ,  $A_1 C$  and  $A_1 D$  intersects the lines  $A_1 B_1$ ,  $A_1 C$  and  $A_1 D$  intersects the lines  $A_1 B_1$ ,  $A_1 C_1$  and  $A_1 D_1$  at the points  $M$ ,  $N$  and  $P$  respectively.

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- (a) Prove that  $M$ ,  $N$  and  $P$  lies on a common line.
- (b) If  $E$  and  $F$  are the feets of the perpendiculars from  $A$  to  $A_1B$  and  $A_1D$ . Prove that the lines  $PE$ ,  $MF$  and  $AN$  have a single point in common.
6. Let  $a$ ,  $b$  and  $c$  are real numbers such that equation  $ax^2+bx+c = 0$  have real roots. Prove that if  $|a(b - c)| > |b^2 - ac| + |c^2 - ab|$  then the equation have at least one root from the interval  $(0, 2)$ .

## Bulgarian Mathematical Olympiad 2002, III Round

*First Day, ? April 2002*

1. Find all triples  $(x, y, z)$  of positive integers such that

$$x! + y! = 15 \cdot 2^{z!}$$

*Nikolai Nikolov, Emil Kolev*

2. Let  $E$  and  $F$  be points on the sides  $AD$  and  $CD$  of a parallelogram  $ABCD$  such that  $\angle AEB = \angle AFB = 90^\circ$ , and  $G$  be the point on  $BF$  for which  $EG \parallel AB$ . If  $H = AF \cap BE$  and  $I = DH \cap BC$ , prove that  $FI \perp GH$ .

*Ivailo Kortezov*

3. Find all positive integers  $n$  for which there exist real numbers  $x$ ,  $y$  and  $z$  such that

$$x = y - \frac{1}{y^n} \quad , \quad y = z - \frac{1}{z^n} \quad , \quad z = x - \frac{1}{x^n}$$

*Sava Grozdev*

*Second day, ? april 2002*

4. Two points  $A$  and  $B$  are given on a circle, and a point  $C$  varies on it so that triangle  $ABC$  is an acute triangle. Let  $E$  and  $F$  be the orthogonal projections of the midpoint of the segment  $AB$  on  $AC$  and  $BC$ . Prove that the perpendicular bisector of the segment  $EF$  passes through a fixed point.

*Alexander Ivanov*

5. Let  $a$ ,  $b$  and  $c$  be positive numbers such that

$$abc \leq \frac{1}{4} \quad \text{and} \quad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} < 9$$

Prove that there exists a triangle with sides  $a$ ,  $b$  and  $c$ .

*Oleg Mushkarov, Nikolai Nikolov*

6. Let  $n \geq 3$  be a positive integer and let  $(a_1, a_2, \dots, a_n)$  be an arbitrary  $n$ -tuple of different real numbers with positive sum. A permutation  $(b_1, b_2, \dots, b_n)$  of these numbers is called *good* if  $b_1 + b_2 + \dots + b_k > 0$  for any  $k = 1, 2, \dots, n$ . Find the least possible number of *good* permutations.

*Aleksander Ivanov*

## 51-th Bulgarian Mathematical Olympiad 2002, IV Round

*First Day, ? april 2002*

1. Let  $\{a_n\}_1^\infty$  be a sequence of real numbers, such that  $a_{n+1} = \sqrt{a_n^2 + a_n} - 1$ . Prove that  $a_1 \notin (-2, 1)$ .

*Oleg Mushkarov, Nikolai Nikolov*

2. Consider the orthogonal projections of the vertices  $A$ ,  $B$  and  $C$  of triangle  $ABC$  on external bisectors of  $\angle ACB$ ,  $\angle BAC$  and  $\angle ABC$ , respectively. Prove that if  $d$  is the diameter of the circumcircle of the triangle, which is formed by the feet of the projections, while  $r$  and  $p$  are the inradius and the semi-perimetr of  $\triangle ABC$ , respectively, then  $r^2 + p^2 = d^2$ .

*Alexander Ivanov*

3. Given are  $n^2$  points in the plane, such that no three of them are collinear, where  $n \geq 4$  is a positive integer of the form  $3k + 1$ . What is the minimal number of connecting segments among the points, such that for each  $n$ -plet of points we can find four points, which are all connected to each other?

*Alexander Ivanov, Emil Kolev*

*Second day, ? april 2002*

4. Let  $I$  be the incenter of a non-equilateral triangle  $ABC$  and  $T_1$ ,  $T_2$ ,  $T_3$  be the tangency points of the incircle with the sides  $BC$ ,  $CA$ ,  $AB$ , respectively. Prove that the orthocenter of  $\triangle T_1T_2T_3$  lies on the line  $OI$ , where  $O$  is the circumcenter of  $\triangle ABC$

*Georgi Ganchev*

5. Find all pairs  $(b, c)$  of positive integers, such that the sequence defined by

$$a_1 = b, \quad a_2 = c \quad \text{and} \quad a_{n+2} = |3a_{n+1} - 2a_n| \quad \text{for } n \geq 1$$

has only finite number of composite terms.

*Oleg Mushkarov, Nikolai Nikolov*

6. Find the smallest number  $k$ , such that  $\frac{\ell_a + \ell_b}{a + b} < k$  for all triangles with sides  $a$  and  $b$  and bisectors  $\ell_a$  and  $\ell_b$  to them, respectively.

*Sava Grodzev, Svetlozar Doichev, Oleg Mushkarov, Nikolai Nikolov*

## Bulgarian Mathematical Olympiad 2003, III Round

*First Day, 19 april 2003*

1. A rectangular trapezium with area 10 and height 4 is divided with a line parallel to its bases on two trapeziums in which can be inscribed circles. Find the radiuses of these circles. (Oleg Mushkarov)
2. It is given a natural number  $n$ . Jana writes natural numbers and then Ivo deletes some of them (zero or more but not all numbers simultaneously) and then before each of the not deleted numbers inserts + or - sign, Ivo wins if the result is divisible by 2003 else Yana wins. Who of them have a winning strategy? (Ivailo Korteov)
3. Find all real numbers  $a$  such that

$$4[an] = n + [a[an]]$$

for every natural number  $n$ . ( $[x]$  is the biggest integer number not greater than  $x$ ). (Nikolai Nikolov)

*Second day, 20 april 2003*

4. The point  $D$  from the side  $AC$  of triangle  $ABC$  is such that:  $BD = CD$ . Through the point  $E$  from the side  $BC$  is drawn a line parallel to  $BD$  intersecting  $AB$  at point  $F$ . If  $G$  is the intersecting point of  $AE$  and  $BD$  prove that:  $\angle BCG = \angle BCF$ . (Oleg Mushkarov, Nikolai Nikolov)
5. Find all real solutions of the system:

$$\begin{cases} x + y + z = 3xy \\ x^2 + y^2 + z^2 = 3xz \\ x^3 + y^3 + z^3 = 3yz \end{cases}$$

(Sava Grozdev, Svetlozar Doichev)

6. We will say that the subset  $C$  consisting of natural numbers is *good* if for each integer number  $k$  there exists  $a, b \in C$ ,  $a \neq b$  such that the numbers  $a + k$  and  $b + k$  aren't relative prime. Prove that if the sum of elements of  $C$  is equal to 2003 then for some  $c \in C$  the set  $C - \{c\}$  is also *good*.

(Aleksander Ivanov, Emil Kolev)



Bulgarian Mathematical Olympiad 2004  
Regional Round, April 17-18

*Grade 9*

1. Find all values of  $a$  such that the equation

$$\sqrt{(4a^2 - 4a - 1)x^2 - 2ax + 1} = 1 - ax - x^2$$

has exactly two solutions. (Sava Grozdev, Svetlozar Doychev)

2. Let  $A_1$  and  $B_1$  be points on the sides  $AC$  and  $BC$  of  $\triangle ABC$  such that  $4 \cdot AA_1 \cdot BB_1 = AB^2$ . If  $AC = BC$ , prove that the line  $AB$  and the bisectors of  $\angle AA_1B_1$  and  $\angle BB_1A_1$  are concurrent. (Sava Grozdev, Svetlozar Doychev)

3. Let  $a, b, c > 0$  and  $a + b + c = 1$ . Prove that

$$\frac{9}{10} \leq \frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} < 1$$

(Sava Grozdev, Svetlozar Doychev)

4. Solve in integers the equation

$$x^3 + 10x - 1 = y^3 + 6y^2.$$

(Sava Grozdev, Svetlozar Doychev)

5. A square  $n \times n$  ( $n > 2$ ) is divided into  $n^2$  unit squares colored in black or white such that the squares at the four corners of any rectangle (containing at least four squares) have no the same color. Find the maximum possible value of  $n$ . (Sava Grozdev, Svetlozar Doychev)

6. Consider the equations

$$[x]^3 + x^2 = x^3 + [x]^2 \quad \text{and} \quad [x^3] + x^2 = x^3 + [x^2]$$

where  $[t]$  is the greatest integer that does not exceed  $t$ . Prove that:

- (a) any solution of the first equation is an integer;
- (b) the second equation has a non-integer solution.

(Sava Grozdev, Svetlozar Doychev)

## Grade 10

1. Solve the inequality

$$\sqrt{x^2 - 1} + \sqrt{2x^2 - 3} + x\sqrt{3} > 0$$

(Peter Boyvalenkov)

2. Let  $M$  be the centroid of  $\triangle ABC$ . Prove that:

$$(a) \cot \angle AMB = \frac{BC^2 + CA^2 - 5AB^2}{12[ABC]};$$

$$(b) \cot \angle AMB + \cot \angle BMC + \cot \angle CMA \leq -\sqrt{3}.$$

(Peter Boyvalenkov)

3. In a school there are  $m$  boys and  $j$  girls,  $m \geq 1$ ,  $1 \leq j < 2004$ . Every student has sent a post card to every student. It is known that the number of the post cards sent by the boys is equal to the number of the post cards sent by girl to girl. Find all possible values of  $j$ .

(Ivailo Korteov)

4. Consider the function

$$f(x) = (a^2 + 4a + 2)x^3 + (a^3 + 4a^2 + a + 1)x^2 + (2aa^2)x + a^2,$$

where  $a$  is a real parameter.

$$(a) \text{ Prove that } f(-a) = 0.$$

$$(b) \text{ Find all values of } a \text{ such that the equation } f(x) = 0 \text{ has three different positive roots.}$$

(Ivan Landjev)

5. Let  $O$  and  $G$  be respectively the circumcenter and the centroid of  $\triangle ABC$  and let  $M$  be the midpoint of the side  $AB$ . If  $OG \perp CM$ , prove that  $\triangle ABC$  is isosceles.

(Ivailo Korteov)

6. Prove that any graph with 10 vertices and 26 edges contains least 4 triangles.

(Ivan Landjev)

*Grade 11*

1. Find all values of  $x \in (-\pi, \pi)$  such that the numbers  $2^{\sin x}$ ,  $2 - 2^{\sin x + \cos x}$  and  $2^{\cos x}$  are consecutive terms of a geometric progression. (Emil Kolev)
2. The lines through the vertices  $A$  and  $B$  that are tangent to circumcircle of an acute  $\triangle ABC$  meet at a point  $D$ . If  $M$  is the midpoint the side  $AB$ , prove that  $\angle ACM = \angle BCD$ . (Emil Kolev)
3. Let  $m \geq 3$  and  $n \geq 2$  be integers. Prove that in a group of  $N = mn - n + 1$  people such that there are two familiar people among any  $m$ , there is a person who is familiar with  $n$  people. Does the statement remain true if  $N < mn - n + 1$ ? (Alexander Ivanov)
4. The points  $D$  and  $E$  lie respectively on the perpendicular bisectors of the sides  $AB$  and  $BC$  of  $\triangle ABC$ . It is known that  $D$  is an interior point for  $\triangle ABC$ ,  $E$  does not and  $\angle ADB = \angle CEB$ . If the line  $AE$  meets the segment  $CD$  at a point  $O$ , prove that the areas of  $\triangle ACO$  and the quadrilateral  $DBEO$  are equal. (Alexander Ivanov)
5. Let  $a$ ,  $b$  and  $c$  be positive integers such that one of them is coprime with any of the other two. Prove that there are positive integers  $x$ ,  $y$  and  $z$  such that  $x^a = y^b + z^c$ . (Alexander Ivanov)
6. One chooses a point in the interior of  $\triangle ABC$  with area 1 and connects it with the vertices of the triangle. Then one chooses a point in the interior of one of the three new triangles and connects it with its vertices, etc. At any step one chooses a point in the interior of one of the triangles obtained before and connects it with the vertices of this triangle. Prove that after the  $n$ -th step:
  - (a)  $\triangle ABC$  is divided into  $2n + 1$  triangles;
  - (b) there are two triangles with common side whose combined area is not less than  $\frac{2}{2n+1}$ .

(Alexander Ivanov)

*Grade 12*

1. Solve in integers the equation

$$2^a + 8b^2 - 3^c = 283.$$

(Oleg Mushkarov, Nikolai Nikolov)

2. Find all values of  $a$  such that the maximum of the function

$$f(x) = \frac{ax - 1}{x^4 - x^2 + 1}$$

is equal to 1. (Oleg Mushkarov, Nikolai Nikolov)

3. A plane bisects the volume of the tetrahedron  $ABCD$  and meets the edges  $AB$  and  $CD$  respectively at points  $M$  and  $N$  such that  $\frac{AM}{BM} = \frac{CN}{DN} \neq 1$ . Prove that the plane passes through the midpoints of the edges  $AC$  and  $BD$ . (Oleg Mushkarov, Nikolai Nikolov)

4. Let  $ABCD$  be a circumscribed quadrilateral. Find  $\angle BCD$  if  $AC = BC$ ,  $AD = 5$ ,  $E = AC \cap BD$ ,  $BE = 12$  and  $DE = 3$ . (Oleg Mushkarov, Nikolai Nikolov)

5. A set  $A$  of positive integers less than 2000000 is called *good* if  $2000 \in A$  and  $a$  divides  $b$  for any  $a, b \in A$ ,  $a < b$ . Find:

- (a) the maximum possible cardinality of a good set;
- (b) the number of the good sets of maximal cardinality.

(Oleg Mushkarov, Nikolai Nikolov)

6. Find all non-constant polynomials  $P(x)$  and  $Q(x)$  with real coefficients such that  $P(x)Q(x+1) = P(x+2004)Q(x)$  for any  $x$ . (Oleg Mushkarov, Nikolai Nikolov)

# Bulgarian Mathematical Olympiad 2005

## Regional Round, April 16-17

### Grade 9

1. Find all values of the real parameters  $a$  and  $b$  such that the remainder in the division of the polynomial  $x^4 - 3ax^3 + ax + b$  by the polynomial  $x^2 + 1$  is equal to  $(a^2 + 1)x + 3b^2$ . (Peter Boyvalenkov)
2. Two tangent circles with centers  $O_1$  and  $O_2$  are inscribed in a given angle. Prove that if a third circle with center on the segment  $O_1O_2$  is inscribed in the angle and passes through one of the points  $O_1$  and  $O_2$  then it passes through the other one too. (Peter Boyvalenkov)

3. Let  $a$  and  $b$  be integers and  $k$  be a positive integer. Prove that if  $x$  and  $y$  are consecutive integers such that

$$a^k x^k y = a,$$

then  $|ab|$  is a perfect  $k$ -th power. (Peter Boyvalenkov)

4. Find all values of the real parameter  $p$  such that the equation  $|x^2 - px + p| = p$  has four real roots  $x_1, x_2, x_3$  and  $x_4$  such that

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 20.$$

(Ivailo Korteov)

5. Let  $ABCD$  be a cyclic quadrilateral with circumcircle  $k$ . The rays  $\overrightarrow{DA}$  and  $\overrightarrow{CB}$  meet at point  $N$  and the line  $NT$  is tangent to  $k$ ,  $T \in k$ . The diagonals  $AC$  and  $BD$  meet at the centroid  $P$  of  $\triangle NTD$ . Find the ratio  $NT : AP$ . (Ivailo Korteov)
6. A card game is played by five persons. In a group of 25 persons all like to play that game. Find the maximum possible number of games which can be played if no two players are allowed to play simultaneously more than once. (Ivailo Korteov)

## Grade 10

1. Solve the system

$$\begin{cases} 3 \cdot 4^x + 2^{x+1} \cdot 3^y 9^y = 0 \\ 2 \cdot 4^x - 5 \cdot 2^x 3^y + 9^y = -8 \end{cases}$$

(Ivan Landjev)

2. Given a quadrilateral  $ABCD$  set  $AB = a$ ,  $BC = b$ ,  $CD = c$ ,  $DA = d$ ,  $AC = e$  and  $BD = f$ . Prove that:

(a)  $a^2 + b^2 + c^2 + d^2 \geq e^2 + f^2$ ;

(b) if the quadrilateral  $ABCD$  is cyclic then  $|a - c| \geq |e - f|$ .

(Stoyan Atanassov)

3. Find all pairs of positive integers  $(m, n)$ ,  $m > n$ , such that

$$[m^2 + mn, mn - n^2] + [mn, mn] = 2^{2005}$$

where  $[a, b]$  denotes the least common multiple of  $a$  and  $b$ .

(Ivan Landjev)

4. Find all values of the real parameter  $a$  such that the number of the solutions of the equation

$$3(5x^2 a^4) - 2x = 2a^2(6x1)$$

does not exceed the number of the solutions of the equation

$$2x^3 + 6x = (3^{6a} - 9) \sqrt{2^{8a} - \frac{1}{6}} - (3a - 1)^2 12^x$$

(Ivan Landjev)

5. Let  $H$  be the orthocenter of  $\triangle ABC$ ,  $M$  be the midpoint of  $AB$  and  $H_1$  and  $H_2$  be the feet of the perpendiculars from  $H$  to the inner and the outer bisector of  $\angle ACB$ , respectively. Prove that the points  $H_1$ ,  $H_2$  and  $M$  are colinear. (Stoyan Atanassov)



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6. Find the largest possible number  $A$  having the following property: if the numbers  $1, 2, \dots, 1000$  are ordered in arbitrary way then there exist 50 consecutive numbers with sum not less than  $A$ .  
(Ivan Landjev)

## Grade 11

1. Find all values of the real parameter  $a$  such that the equation

$$a(\sin 2x + 1) + 1 = (a+3)(\sin x + \cos x)$$

has a solution. (Emil Kolev)

2. On the sides of an acute  $\triangle ABC$  of area 1 points  $A_1 \in BC$ ,  $B_1 \in CA$  and  $C_1 \in AB$  are chosen so that

$$\angle CC_1B = \angle AA_1C = \angle BB_1A = \phi,$$

where the angle  $\phi$  is acute. The segments  $AA_1$ ,  $BB_1$  and  $CC_1$  meet at points  $M$ ,  $N$  and  $P$ .

- (a) Prove that the circumcenter of  $\triangle MNP$  coincides with the orthocenter of  $\triangle ABC$ .  
 (b) Find  $\phi$ , if  $[MNP] = 2 - \sqrt{3}$ .

(Emil Kolev)

3. Let  $n$  be a fixed positive integer. The positive integers  $a$ ,  $b$ ,  $c$  and  $d$  are less than or equal to  $n$ ,  $d$  is the largest one and they satisfy the equality

$$(ab + cd)(bc + ad)(ac + bd) = (da)^2(db)^2(dc)^2.$$

- (a) Prove that  $d = a + b + c$ .  
 (b) Find the number of the quadruples  $(a, b, c, d)$  which have the required properties.

(Alexander Ivanov)

4. Find all values of the real parameter  $a$  such that the equation

$$\log_{ax}(3^x + 4^x) = \log_{(ax)^2}(7^2(4^x - 3^x)) + \log_{(ax)^3} 8^{x-1}$$

has a solution. (Emil Kolev)

5. The bisectors of  $\angle BAC$ ,  $\angle ABC$  and  $\angle ACB$  of  $\triangle ABC$  meet its circumcircle at points  $A_1$ ,  $B_1$  and  $C_1$ , respectively. The side  $AB$  meets the lines  $C_1B_1$  and  $C_1A_1$  at points  $M$  and  $N$ , respectively, the side  $BC$  meets the lines  $A_1C_1$  and  $A_1B_1$  at points  $P$  and  $Q$ , respectively, and the side  $AC$  meets the lines  $B_1A_1$  and  $B_1C_1$  at points  $R$  and  $S$ , respectively. Prove that:
- (a) the altitude of  $\triangle CRQ$  through  $R$  is equal to the inradius of  $\triangle ABC$ ;
  - (b) the lines  $MQ$ ,  $NR$  and  $SP$  are concurrent.

(Alexander Ivanov)

6. Prove that amongst any 9 vertices of a regular 26-gon there are three which are vertices of an isosceles triangle. Do there exist 8 vertices such that no three of them are vertices of an isosceles triangle?

(Alexander Ivanov)

## Grade 12

1. Prove that if  $a$ ,  $b$  and  $c$  are integers such that the number

$$\frac{a(a) + b(b) + c(c)}{2}$$

is a perfect square, then  $a = b = c$ . (Oleg Mushkarov)

2. Find all values of the real parameters  $a$  and  $b$  such that the graph of the function  $y = x^3 + ax + b$  has exactly three common points with the coordinate axes and they are vertices of a right triangle. (Nikolai Nikolov)

3. Let  $ABCD$  be a convex quadrilateral. The orthogonal projections of  $D$  on the lines  $BC$  and  $BA$  are denoted by  $A_1$  and  $C_1$ , respectively. The segment  $A_1C_1$  meets the diagonal  $AC$  at an interior point  $B_1$  such that  $DB_1 \geq DA_1$ . Prove that the quadrilateral  $ABCD$  is cyclic if and only if

$$\frac{BC}{DA_1} + \frac{BA}{DC_1} = \frac{AC}{DB_1}$$

(Nikolai Nikolov)

4. The point  $K$  on the edge  $AB$  of the cube  $ABCD A_1 B_1 C_1 D_1$  is such that the angle between the line  $A_1 B$  and the plane  $(B_1 C K)$  is equal to  $60^\circ$ . Find  $\tan \alpha$ , where  $\alpha$  is the angle between the planes  $(B_1 C K)$  and  $(ABC)$ . (Oleg Mushkarov)
5. Prove that any triangle of area  $\sqrt{3}$  can be placed into an infinite band of width  $\sqrt{3}$ . (Oleg Mushkarov)
6. Let  $m$  be a positive integer,  $A = \{-m, -m + 1, \dots, m - 1, m\}$  and  $f : A \rightarrow A$  be a function such that  $f(f(n)) = -n$  for every  $n \in A$ .

- (a) Prove that the number  $m$  is even.
- (b) Find the number of all functions  $f : A \rightarrow A$  with the required property.

(Nikolai Nikolov)

# Bulgarian Mathematical Olympiad 2006

## Regional Round, April 15-16

### Grade 9

1. Find all real numbers  $a$  such that the roots  $x_1$  and  $x_2$  of the equation

$$x^2 + 6x + 6a - a^2 = 0$$

satisfy the relation  $x_2 = x_1^3 - 8x_1$ . (Ivan Landjev)

2. Two circles  $k_1$  and  $k_2$  meet at points  $A$  and  $B$ . A line through  $B$  meets the circles  $k_1$  and  $k_2$  at points  $X$  and  $Y$ , respectively. The tangent lines to  $k_1$  at  $X$  and to  $k_2$  at  $Y$  meet at  $C$ . Prove that:

(a)  $\angle XAC = \angle BAY$ .

(b)  $\angle XBA = \angle XBC$ , if  $B$  is the midpoint of  $XY$ .

(Stoyan Atanasov)

3. The positive integers  $\ell, m, n$  are such that  $m - n$  is a prime number and

$$8(\ell^2 - mn) = 2(m^2 + n^2) + 5(m + n)\ell$$

Prove that  $11\ell + 3$  is a perfect square. (Ivan Landjev)

4. Find all integers  $a$  such that the equation

$$x^4 + 2x^3 + (a^2 - 9)x^2 - 4x + 4 = 0$$

has at least one real root. (Stoyan Atanasov)

5. Given a right triangle  $ABC$  ( $\angle ACB = 90^\circ$ ), let  $CH$ ,  $H \in AB$ , be the altitude to  $AB$  and  $P$  and  $Q$  be the tangent points of the incircle of  $\triangle ABC$  to  $AC$  and  $BC$ , respectively. If  $AQ \perp HP$  find the ratio  $\frac{AH}{BH}$ .

(Stoyan Atanasov)

6. An air company operates 36 airlines in a country with 16 airports. Prove that one can make a round trip that includes 4 airports. (Ivan Landjev)

## Grade 10

1. A circle  $k$  is tangent to the arms of an acute angle  $AOB$  at points  $A$  and  $B$ . Let  $AD$  be the diameter of  $k$  through  $A$  and  $BP \perp AD$ ,  $P \in AD$ . The line  $OD$  meets  $BP$  at point  $M$ . Find the ratio  $\frac{AH}{BH}$ .

(Peter Boyvaienkov)

2. Find the maximum of the function

$$f(x) = \frac{\log x \log x^2 + \log x^3 + 3}{\log^2 x + \log x^2 + 2}$$

and the values of  $x$ , when it is attained. (Ivailo Kortezov)

3. Let  $\mathbb{Q}^+$  be the set of positive rational numbers. Find all functions  $f : \mathbb{Q}^+ \rightarrow \mathbb{R}$  such that  $f(1) = 1$ ,  $f(1/x) = f(x)$  for any  $x \in \mathbb{Q}^+$  and  $xf(x) = (x+1)f(x-1)$  for any  $x \in \mathbb{Q}^+$ ,  $x > 1$ . (Ivailo Kortezov)

4. The price of a merchandise dropped from March to April by  $x\%$ , and went up from April to May by  $y\%$ . It turned out that in the period from March to May the price dropped by  $(y-x)\%$ . Find  $x$  and  $y$  if they are positive integers (the price is positive for the whole period). (Ivailo Kortezov)

5. Let  $ABCD$  be a parallelogram such that  $\angle BAD < 90^\circ$  and  $DE$ ,  $E \in AB$ , and  $DF$ ,  $F \in BC$ , be the altitudes of the parallelogram. Prove that

$$4(AB \cdot BC \cdot EF + BD \cdot AE \cdot FC) \leq 5 \cdot AB \cdot BC \cdot BD.$$

Find  $\angle BAD$  if the equality occurs. (Ivailo Kortezov)

6. See problem 6 (grade 9).

## Grade 11

1. Let  $k$  be a circle with diameter  $AB$  and let  $C \in k$  be an arbitrary point. The excircles of  $\triangle ABC$  tangent to the sides  $AC$  and  $BC$  are tangent to the line  $AB$  at points  $M$  and  $N$ , respectively. Denote by  $O_1$  and  $O_2$  the circumcenters of  $\triangle AMC$  and  $\triangle BNC$ . Prove that the area of  $\triangle O_1CO_2$  does not depend on  $C$ .  
(Alexander Ivanov)

2. Prove that

$$t^2(xy + yz + zx) + 2t(x + y + z) + 3 \geq 0 \quad \text{for all } x, y, z, t \in [0, 1]$$

(Nikolai Nikolov)

3. Consider a set  $S$  of 2006 points in the plane. A pair  $(A, B) \in S \times S$  is called *isolated* if the disk with diameter  $AB$  does not contain other points from  $S$ . Find the maximum number of *isolated* pairs.  
(Alexander Ivanov)

4. Find the least positive integer  $a$  such that the system

$$\begin{cases} x + y + z = a \\ x^3 + y^3 + z^3 = a \end{cases}$$

has no an integer solution. (Oleg Mushkarov)

5. The tangent lines to the circumcircle  $k$  of an isosceles  $\triangle BAC$ ,  $AC = BC$ , at the points  $B$  and  $C$  meet at point  $X$ . If  $AX$  meets  $k$  at point  $Y$ , find the ratio  $AY/BY$ .  
(Emil Kola)

6. Let  $a_1, a_2, \dots$  be a sequence of real numbers less than 1 and such that  $a_{n+1}(a_n + 2) = 3$ ,  $n \geq 1$ . Prove that:

(a)  $-\frac{7}{2} < a_n < -2$ ;

(b)  $a_n = -3$  for any  $n$ .

(Nikolai Nikolov)

*Grade 12*

1. Find the area of the triangle determined by the straight line with equation  $x - y + 1 = 0$  and the tangent lines to the graph of the parabola  $y = x^2 - 4x + 5$  at its common points with the line.  
(Emil Kolev)

2. See problem n.5 (grade 11).

3. Find all real numbers  $a$ , such that the inequality

$$x^4 + 2ax^3 + a^2x^2 - 4x + 3 > 0$$

(Nikolai Nikolov)

4. Find all positive integers  $n$  for which the equality

$$\frac{\sin(n\alpha)}{\sin \alpha} - \frac{\cos(n\alpha)}{\cos \alpha} = n - 1$$

holds true for all  $a \neq \frac{k\pi}{2}$ ,  $k \in \mathbb{Z}$ . (Emil Kolev)

5. A plane intersects a tetrahedron  $ABCD$  and divides the medians of the triangles  $DAB$ ,  $DBC$  and  $DCA$  through  $D$  in ratios  $1 : 2$ ,  $1 : 3$  and  $1 : 4$  from  $D$ , respectively. Find the ratio of the volumes of the two parts of the tetrahedron cut by the plane. (Oleg Mushkarov)

6. See problem 6 (grade 11).



Bulgarian Mathematical Olympiad 2007  
Regional Round, April 14-15

*Grade 9*

*First Day*

1. Find the real solutions of the equation:

$$\sqrt{x-3} + \sqrt{7-x} = x^2 - 10x + 23$$

2. The quadrilateral  $ABCD$  that have no two parallel sides is inscribed in a circle with radii 1. The point  $E$  belongs to  $AB$  and  $F$  belongs to  $CD$  are such that  $CE \parallel AD$  and  $CF \parallel AB$ . It is also known that the circumscribed circle around the triangle  $CDF$  intersects for second time the diagonal  $AC$  in an internal point. Find the biggest possible value of the expression:  $AB \cdot AE + AD \cdot AF$ .
3. Find the smallest possible value of the expression:

$$M = x + \frac{y^2}{9x} + \frac{3z^2}{32y} + \frac{2}{z}$$

*Second day*

4. Prove that there doesn't exist values for a real parameter  $a$  for which the system:

$$\begin{cases} x^2 = x + ay + 1 \\ y^2 = ax + y + 1 \end{cases}$$

have exactly three different solutions.

5. Find all even natural numbers  $n$  and all real numbers  $a$ , for which the remainder of division of the polynomial  $x^n - x^{n-1} + ax^4 + 1$  by the polynomial  $x^2 - a^2$  is equal to  $97 - (a + 14)x$ .

6. It is given a regular 16-gon  $A_1 \cdots A_{16}$  which vertices lie on a circle (circumference) with center  $O$ . Is it possible to be chosen some vertices of the 16-gon in such a way when we rotate the 16-gon around  $O$  by angles:  $\frac{360^\circ}{16}$ ,  $2 \cdot \frac{360^\circ}{16}$ ,  $\dots$ ,  $16 \cdot \frac{360^\circ}{16}$  the segments connecting the vertices chosen to all the sides and diagonals of the 16-gon exactly two times.

Bulgarian Mathematical Olympiad 2007  
Regional Round, April 14-15

*Grade 10*

*First Day*

1. Solve the equation inequality:

$$\frac{\sqrt{-x^2 + 6x + 1} + 5x - 15}{2x - 5} \geq 2$$

2. It is given a triangle  $ABC$ .  $D$ ,  $E$ ,  $F$  are the tangent points of externally inscribed circles to the sides  $BC$ ,  $CA$ ,  $AB$  respectively.

- (a) Prove that  $AD$ ,  $BE$  and  $CF$  intersects at a common point.  
(b) Prove that if the common point of  $AD$ ,  $BE$  and  $CF$  lies on the incircle of the triangle, then the perimeter of the triangle is four times greater than its smallest side.

3. The natural numbers  $a_1, a_2, \dots, a_n$ ,  $n \geq 3$  are such that:

$$b_1 = \frac{a_n + a_2}{a_1}, \quad b_2 = \frac{a_1 + a_2}{a_2}, \quad \dots, \quad b_n = \frac{a_{n-1} + a_1}{a_n}$$

are integer numbers. Prove that  $b_1 + b_2 + \dots + b_n \leq 3n - 1$ .

*Second day*

4. Find the values of the real paramether  $a$ , for which the equation

$$\log_{x-a}(x+a) = 2$$

have only one solution.

5. Find the count of all pairs of natural numbers  $(m, n)$  which are solutions of the system:

$$\begin{cases} 47^m - 48^n + 1 \equiv 0 \pmod{61} \\ 3m + 2n = 1000 \end{cases}$$

have exactly three different solutions.

6. It is given a regular 16-gon  $A_1 \cdots A_{16}$  which vertices lie on a circle (circumference) with center  $O$ . Is it possible to be chosen some vertices of the 16-gon in such a way when we rotate the 16-gon around  $O$  by angles:  $\frac{360^\circ}{16}, 2 \cdot \frac{360^\circ}{16}, \dots, 16 \cdot \frac{360^\circ}{16}$  the segments connecting the vertices chosen to all the sides and diagonals of the 16-gon exactly two times.

Bulgarian Mathematical Olympiad 2007  
Regional Round, April 14-15

*Grade 11*

*First Day*

1. Find all real values of the parameter  $a$  for which the system:

$$\begin{cases} \sin x + \cos y = 4a + 6 \\ \cos x + \sin y = 3a + 2 \end{cases}$$

have a solution.

2. Let  $n$  is a natural number and  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$  are positive numbers. Prove the inequality:

$$(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n) + 2^{n-1} \left( \frac{1}{a_1 b_1} + \frac{1}{a_2 b_2} + \cdots + \frac{1}{a_n b_n} \right) \geq 2^{n-1}(a+2)$$

When does equality holds?

3. For the set  $A$  composed from real numbers we denote with  $A^+$  the count of different numbers can be achieved as a sum of two (not necessary different) numbers and  $A^-$  is the count of different positive numbers that can be achieved as a difference between two numbers from  $A$ . Let  $A$  is such 2007-element set for which  $A^+$  have maximal possible value. Find  $A^-$ .

*Second day*

4. It is given the sequence  $\{a_n\}_{n=1}^{\infty}$  defined with the equalities:  $a_1 = 4$ ,  $a_2 = 3$  and  $2a_{n+1} = 3a_n - a_{n-1}$  for  $n \geq 2$ .
- (a) Prove that the sequence converges and find its limit.
- (b) Calculate the following limit:

$$\lim_{n \rightarrow \infty} \frac{(a_n - 1)(a_n - 2)}{(\sqrt{a_n + 2} - 2)(\sqrt{12a_n - 8} - 2)}$$

5. Let the point  $D$  is a point from the side  $AB$  of the triangle  $ABC$ . We denote with  $P$  and  $M$  respectively the incenters of the triangles  $ADC$  and  $BDC$ .  $Q$  and  $N$  are the centers of externally inscribed circles in  $ADC$  to the side  $AD$  and in  $BDC$  to the side  $BD$  respectively. Let  $K$  and  $L$  are the symmetric points of  $Q$  and  $N$  to the line  $AB$ .
- (a) Prove that the lines  $AB$ ,  $QN$  and  $PM$  intersects at a common point or they are parallel.
- (b) Prove that if the points  $M$ ,  $P$ ,  $K$  and  $L$  lies on a common circle, then  $ABC$  is isosceles.
6. Let  $a$  and  $b$  are natural numbers and  $a = 4k + 3$  for some integer number  $k$ . Prove that if the equation:

$$x^2 + (a - 1)y^2 + az^2 = b^n$$

have solutions in positive integers  $x$ ,  $y$ , and  $z$  for  $n = 1$  then the equation have positive solution for every natural number  $n$ .

Bulgarian Mathematical Olympiad 2007  
Regional Round, April 14-15

*Grade 12*

*First Day*

1. Find all real values of the parameter  $a$  for which there exists two mutually perpendicular tangents to the function graph:

$$f(x) = ax + \sin x.$$

2. In sphere with radii 2 is inscribed a pyramid  $ABCD$  for which  $AB = 2$ ,  $CD = \sqrt{7}$  and  $\angle ABC = \angle BAD = 90^\circ$ . Find the angle between the lines  $AD$  and  $BC$ .
3. Prove that if  $x$  and  $y$  are integer numbers, then the number  $x^2(y - 2) + y^2(x - 2)$  is not prime.

*Second day*

4. Let  $I$  is the center of the externally inscribed circle tangent to the side  $AB$  of the triangle  $ABC$ .  $S$  is the symmetric point of  $I$  with respect to  $AB$ . The line through  $S$ , perpendicular to  $BI$ , intersects the line  $AI$  at the point  $T$ . Prove that  $CI = CT$ .
5. Solve the equation:

$$\frac{8^x - 2^x}{6^x - 3^x} = 2$$

6. Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(0) \leq 0$  and  $f(x + y) \leq x + f(f(x))$  for any two numbers  $x, y$  from  $\mathbb{R}$ .

56-th Bulgarian Mathematical Olympiad 2007, National  
Round

*First Day, 12-th May 2007*

1. In the quadrilateral  $ABCD$  for which  $\angle BAD < \angle ADC < 180^\circ$  is inscribed a circle with a center  $I$ . Through  $I$  is drawn a line who intersects the sides  $AB$  and  $CD$  respectively at the points  $X$  and  $Y$  in such a way that  $IX = IY$ . Prove that  $AX \cdot DY = BX \cdot CY$ .
2. Find the biggest natural number  $n$  for which we can choose 2007 different numbers from the interval  $[2 \cdot 10^{n-1}, 10^n)$  in such a way that for every two natural numbers  $i$  and  $j$  for which  $1 \leq i < j \leq n$  there exists a number  $\overline{a_1 a_2 \cdots a_n}$  such that  $a_j \geq a_i + 2$ .
3. Find the smallest natural number  $n$  for which  $\cos \frac{\pi}{n}$  cannot be expressed in the form  $p + \sqrt{q} + \sqrt[3]{r}$ , where  $p, q$  and  $r$  are rational numbers.

*Second day, 13-th May 2007*

4. Let  $k, k > 1$  is a given natural number. We call that the set  $S$  of natural numbers is *good* if all natural numbers can be colored in  $k$  colors in such a way that there isn't exist a number from  $S$  that can be expressed as a sum of two different natural numbers, colored in the same color. Find the biggest number  $t$  for which the set:

$$S = \{a + 1, a + 2, a + 3, \dots, a + t\}$$

is *good* for every natural number  $a$ .

5. Find the smallest number  $m$  such that we can cover an equilateral triangle with area 1 using 5 equilateral triangles which sum of areas is equal to  $m$ .
6. Let  $f(x)$  is a polynomial which power is an even number with integer coefficients and its first coefficient is equal to 1. It is known that there exist infinitely many integer numbers  $x$ , for which  $f(x)$  is a square of natural number. Prove that there exists a polynomial  $g(x)$  with integer coefficients for which  $f(x) = g^2(x)$ .



Bulgarian Mathematical Olympiad 2008  
Regional Round, April 19-20

*Grade 9*

*First Day*

1. Find all pairs of integer numbers  $(p, q)$  such that the roots of the equation:

$$(px - q)^2 + (qx - p)^2 = x$$

are also integer numbers.

2. It is given a rhombus  $ABCD$  with sidelength  $a$ . On the line  $AC$  are chosen the points  $M$  and  $N$  in such a way that  $C$  lies between  $A$  and  $N$  and  $MA \cdot NC = a^2$ . We denote with  $P$  the intersection point of  $MD$ ,  $BC$  and  $Q$  is the intersection point of  $ND$ ,  $AB$ . Prove that  $D$  is the incenter of the triangle  $PQB$ .
3. Find all natural numbers  $n$  with exactly 8 natural divisors which sum is equal to 3780 (including 1 and  $n$ ).

*Second day*

4. It is given the isosceles triangle  $ABC$  ( $AC = BC$ ) where the angle  $\angle ACB$  is equal to  $30^\circ$ . The point  $M$  is symmetric to the vertex  $A$  with respect to the line  $BC$ .  $N$  is symmetric to the  $M$  with respect to the vertex  $C$ . If  $P$  is the intersecting point of the lines  $AC$  and  $BN$  and  $Q$  is the intersecting point of the lines  $AN$  and  $PM$  find the ratio  $AQ : QN$ .
5. Solve the equation:

$$x^2 - 13[x] + 11 = 0$$

6. The cities in a country are connected with paths. It is known that two cities are connected with no more than one path(s) and each city is connected with not less than three paths. A traveler left some city must pass through at least six other cities (it is not allow passing a path more than once) before he goes back to its starting position. Prove that the country have at least 24 cities.

Bulgarian Mathematical Olympiad 2008  
Regional Round, April 19-20

*Grade 10*

*First Day*

1. Find all pairs real values of  $x$  such that the following inequalities are satisfied:

$$1 \leq \sqrt{x+2} - \frac{1}{\sqrt{x+2}} \leq 4$$

2. The points  $A$ ,  $B$  and  $C$  are situated on the circumference  $k$  in such a way that the tangents to  $k$  at the points  $A$  and  $B$  intersects at the point  $P$  and  $C$  lies on the bigger arc  $AB$ . Let the line through  $C$  which is perpendicular to  $PC$  intersects the line  $AB$  at the point  $Q$ . Prove that:
- (a) if the lines  $PC$  and  $QC$  intersects  $k$  for second time at the points  $M$  and  $N$  then the angles  $\angle CQP$  and  $\angle CMN$  are equal.
  - (b) If  $S$  is the middle point of  $PQ$  then  $SC$  is tangent to  $k$ .
3. Prove that there exist a function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , such that  $f(f(n)) = 3n$  for all natural numbers  $n$ .

*Second day*

4. It is given the equation:

$$8^x - 2m2^x + 1 + 4m = 8$$

where  $m$  is a real parameter.

- (a) Solve the equation for  $m = 6$ .
- (b) Find all real values of  $m$  for which the equation have exactly one positive solution.

5. If  $a, b, c$  are positive real numbers prove the inequality:

$$\frac{a^5}{bc} + \frac{b^5}{ca} + \frac{c^5}{ab} + \frac{3}{2a^2b^2c^2} \geq 2(a^3 + b^3 + c^3) + \frac{9}{2} - 6abc$$

When does equality holds?

6. An isosceles trapezium with bases 1 and 5 and equal sides with length  $\sqrt{7}$  is covered with 10 circles with radii  $r$ . Prove that  $r \geq 1/2$ . (Here *covered* means each trapezium's point is in at least one of the circles).

Bulgarian Mathematical Olympiad 2008  
Regional Round, April 19-20

*Grade 11*

*First Day*

1. It is given an arithmetic progression  $a_1, a_2, \dots, a_n, \dots$  for which  $a_1 \cdot a_2 < 0$  and

$$(a_1 + a_2 + a_3) \left( \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right) = -\frac{3}{8}$$

Find the smallest natural number  $n > 2$  for which  $\frac{a_n}{a_2}$  is a square of natural number.

2. In the equilateral triangle  $ABC$  is chosen a point  $O$ . Its symmetric point with respect to the sides  $BC$ ,  $CA$  and  $AB$  are denoted respectively with  $A_1$ ,  $B_1$  and  $C_1$ . Prove that the lines  $AA_1$ ,  $BB_1$  and  $CC_1$  intersects at a common point.
3. Let  $a$  and  $b$  are natural numbers. Prove that the sequence:  $\{a_n\}_{n=1}^{\infty}$ , defined with the equalities  $a_1 = a$  and  $a_{n+1} = \phi(an+b)$ ,  $n > 1$  is bounded. For every natural number  $k$  with  $\phi(k)$  is denoted the number of natural numbers smaller than  $k$  and coprime with  $k$ .

*Second day*

4. In a circle with radii  $R = 65$  is inscribed a quadrilateral  $ABCD$  for which  $AB = 50$ ,  $BC = 104$  and  $CD = 120$ . Find the length of the side  $AD$ .
5. (a) It is given the sequence  $a_n = \sqrt[n]{n}$ ,  $n = 1, 2, \dots$ . Prove that

$$\lim_{n \rightarrow \infty} a_n = 1$$

- (b) Let  $f(x)$  is a polynomial with positive integer coefficients. Prove that the sequence  $b_n = \sqrt[n]{f(n)}$ ,  $n = 1, 2, \dots$ , converges and find its limit.
6. Let  $k$  is a natural number. We denote with  $f(k)$  the biggest natural number for which there exists a set  $M$  from natural numbers with  $f(k)$  elements, such that:
- (i) Each element from  $M$  is a divisor of  $k$ .
  - (ii) There are no element from  $M$  that divides some other element from  $M$ .

Prove that if  $m$  and  $n$  are coprime numbers then:

$$f(n \cdot 2^n) \cdot f(m \cdot 2^m) = f(mn \cdot 2^{m+n})$$

# Bulgarian Mathematical Olympiad 2008

## Regional Round, April 19-20

*Grade 12*

*First Day*

1. In a triangle pyramid  $ABCD$  adjacent edges  $DB$  and  $DC$  are equal and  $\angle DAB = \angle DAC$ . Find the volume of the pyramid if  $AB = 15$ ,  $BC = 14$ ,  $CA = 13$  and  $DA = 18$ .
2. Find the values of real parameter  $a$  for which the graphs of the functions  $f(x) = x^2 + a$  and  $g(x) = x^3$  have exactly one common tangent.
3. Is there exists a natural number  $n$  for which the number  $\left(\frac{2008}{n}\right)^3 + \frac{2008}{n}$  is a square of a natural number.

*Second day*

4. Find all natural numbers  $a$  such that

$$\left[ \sqrt{n} + \sqrt{n+1} \right] = \left[ \sqrt{4n+a} \right]$$

for any natural number  $n$  (with  $[x]$  is denoted the integer part of the number  $x$ ).

5. The incircle of  $ABC$  is tangent to the sides  $BC$ ,  $CA$  and  $AB$  respectively at the points:  $A_1$ ,  $B_1$  and  $C_1$ . It is also known that the line  $A_1B_1$  pass through the middle point of the segment  $CC_1$ . Find the angles of the triangle if their sines form an arithmetic progression.
6. Let  $\mathbb{R}$  be the set of all real numbers. Find all functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(x + y^2) \geq (y + 1)f(x)$$

for all  $x, y$  that belongs to  $\mathbb{R}$ .

57-th Bulgarian Mathematical Olympiad 2008, National Round

*First Day, 17-th May 2008*

1. Let  $ABC$  is acute-angled triangle and  $CL$  is its internal angle bisector and  $L \in AB$ . The point  $P$  belongs to the segment  $CL$  in such a way that  $\angle APB = \pi - \frac{1}{2}\angle ACB$ . Let  $k_1$  and  $k_2$  are the circumcircles of  $\triangle APC$  and  $\triangle BPC$ .  $BP \cap k_1 = Q$  and  $BP \cap k_2 = R$ . The tangents to  $k_1$  in  $Q$  and to  $k_2$  in  $B$  intersects at the point  $S$  and the tangents to  $k_1$  at  $R$  and to  $k_2$  at  $A$  intersects at the point  $T$ . Prove that  $AS = BT$ .
2. Are there exists 2008 non-intersecting arithmetic progressions composed from natural numbers such that each of them contains a prime number greater than 2008 and the numbers that doesn't belongs to (some of) the progressions are finite number?
3. Let  $n \in \mathbb{N}$  and  $0 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n \leq \pi$  and  $b_1, b_2, \dots, b_n$  are real numbers for which the following inequality is satisfied:

$$\left| \sum_{i=1}^n b_i \cos(k\alpha_i) \right| < \frac{1}{k}$$

for all  $k \in \mathbb{N}$ . Prove that  $b_1 = b_2 = \dots = b_n = 0$ .

*Second day, 18-th May 2008*

4. Find the smallest natural number  $k$  for which there exists natural numbers  $m$  and  $n$  such that  $1324 + 279m + 5^n$  is  $k$ -th power of some natural number.
5. Let  $n$  is a fixed natural number. Find all natural numbers  $m$  for which

$$\frac{1}{a^n} + \frac{1}{b^n} \geq a^m + b^m$$

is satisfied for every two positive numbers  $a$  and  $b$  with sum equal to 2.

6. Let  $M$  is the set of the integer numbers from the range  $[-n, n]$ . The subset  $P$  of  $M$  is called *base subset* if every number from  $M$  can be expressed as a sum of some different numbers from  $P$ . Find the smallest natural number  $k$  such that every  $k$  numbers that belongs to  $M$  form a *base subset*.