SOUTH AFRICA AND THE 36TH INTERNATIONAL MATHEMATICAL OLYMPIAD

GRAEME WEST

JOHN WEBB

- TALENT SEARCH PROBLEMS
- IMO PROBLEMS
- SOLUTIONS

CANADA 1995

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95
INTERNATIONAL MATHEMATICAL OLYMPIAD

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Preface

The programme to identify, select and train South African teams for the International Mathematical Olympiad (IMO) began in 1991, when South Africa was invited to send an Observer to the IMO in Sweden. Under the auspices of the South African Mathematical Society (SAMS), a nationwide Talent Search was launched, in the form of a correspondence programme in problem-solving, with Mathematical Camps serving as the means of selecting the teams. The first South African IMO team took part in the Olympiad in Moscow in 1992.

The SAMS IMO programme has now settled down. Apart from the Talent Search and the Camps, publications play an important role. To date, five booklets in the SAMS Olympiad Training Notes have appeared, as well as annual reports of the Talent Search and IMO for 1992, 1993 and 1994. This book is a comprehensive account of the Talent Search, camps and training programme for the 1995 IMO.

The Talent Search for a South African team to go to the 1995 IMO in Canada began at the beginning of 1994, and at the end of that year the emerging front runners were invited to attend a camp at the University of Stellenbosch. During the early months of 1995, the students were challenged with progressively more difficult problems, and in April 1995 the survivors of this process were invited to a camp hosted by Rhodes University.

After five days of intensive training and testing at the Rhodes Camp the team was announced. With the IMO scheduled for July, the months of May and June saw the team keeping up the pressure. At the end of June the team gathered for a five-day camp at the University of Witwatersrand, writing a full IMO-style paper each day.

From Wits the team flew to the United States, where they enjoyed a few days sightseeing in New York. The next stop was Kent State University in Ohio for a final training programme. And, at last, they got to Toronto, and took part, with 406 contestants from 72 other countries, in the 35th IMO.

Much can be written about South Africa's participation in the IMO: the stimulation and the challenge of matching wits with the best in the world, the triumphs and the disappointments, sightseeing and friendships. This book tells only the mathematical story, but in as much detail as possible. The Talent Search problems, the tests written at Stellenbosch, Rhodes, Wits and Kent, and the IMO itself, are given, with full solutions.

The purpose of this book is to show future IMO teams the way to success. It is a book to be used as a source of challenging problems by students hoping to get into future teams, and for teachers looking for unusual material to challenge their promising students. The problems in this book can seldom be solved on sight, but require time and concentration. Only after a serious attempt has been made on a problem should the solution be consulted, and then only briefly, to get the idea, before going back to solving the problem independently. At the end, the solution should be carefully studied, and should be regarded as a challenge to the reader to provide better, briefer and more insightful arguments.

The Talent Search problems and solutions were compiled by Professor John Webb, and the Stellenbosch, Rhodes, Witwatersrand and Kent State University camp tests were put together by Professor John Webb, Professor Nic Heideman, Dr Louis le Riche, Professor Valentin Goranko and Dr Graeme West.

Though this book is a joint effort, the bulk of the work was done by Graeme West. He selected, devised, revised or adapted most of the problems, wrote the solutions and was responsible for the typesetting.

Moreover, as Deputy Leader of the South African team, Graeme was responsible for the final training programme, organizing all aspects of the Wits and Kent Camps, sorting out unexpected visa and travel problems and preparing the team, both mathematically and psychologically, for the toughest intellectual challenge they had ever faced. The success that South African teams have achieved in the IMO is substantially Graeme's.

Thanks are due to the University of Stellenbosch, Rhodes University, the University of the Witwatersrand and Kent State University, Ohio, for hosting the Camps, and to the University of Cape Town for financial support for the whole programme.

John Webb

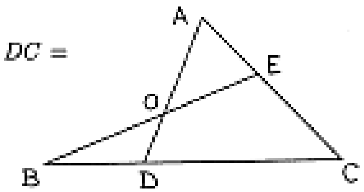
Contents

1 The Talent Search: 1994

1.1 Prove that, if $a^2+b^2+c^2=1$, then $(a-b)^2+(b-c)^2+(c-a)^2 \leq 3$.

1.2 Prove that there are no positive integers m and n such that $2(m^2+mn+n^2)$ is a perfect square.

1.3 Determine $AO:OD$ if $BD:DC = 4:7$ and $AE:EC = 2:3$.



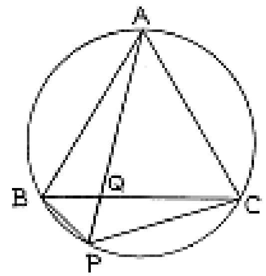
1.4 Regular polygons with m and n sides are inscribed in the same circle. The ratio of their areas is $m:n$. Find all possible values of m and n.

1.5 What is the largest even natural number which cannot be expressed as the sum of two odd composite natural numbers?

2.1 What is the largest positive integer which cannot be written in the form $5x + 7y$, with x and y positive integers?

2.2 In the figure, $\triangle ABC$ is equilateral.
Prove that

$$\frac{1}{PB} + \frac{1}{PC} = \frac{1}{PQ}.$$



2.3 Solve

$$\begin{aligned} \sqrt{2x-1} + \sqrt{y+3} &= 3 \\ 2xy + 6x - y &= 7. \end{aligned}$$

2.4 How many strings are there of 5 digits which do not contain consecutive zeros?

2.5 Prove that $4^n + 2$ is divisible by 6 for every positive integer n.

3.1 Between 5am and 6am the hands of a clock are at right angles exactly twice. How many minutes elapse between these times?

3.2 Prove that $7^{\sqrt{5}} > 5^{\sqrt{7}}$ without using any calculating aid.

3.3 Find all functions f such that

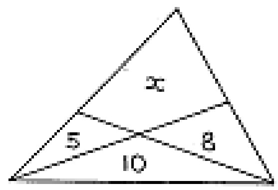
$$f(x)f(y) - f(xy) = x + y$$

for all real values of x and y.

3.4 Prove that among any six integers there will be a pair whose sum or difference is divisible by 9.

3.5 In $\triangle ABC$, $\angle A = 75°$ and $AB = 2CH$, where CH is an altitude. Determine $\angle B$.

4.1 A triangle is divided into four regions of areas 5, 8, 10 and x, as shown. Determine x.



4.2 Find all integers m and n such that $m^3 - n^3 - 6mn = 8$.

4.3 Let a, b, c be three integers, none of which is a multiple of 5. Prove that at least one of $a^2 - b^2$, $b^2 - c^2$, $a^2 - c^2$ is a multiple of 5.

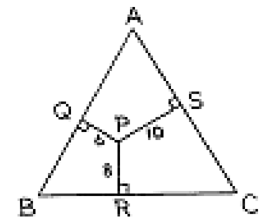
4.4 Prove that, if a, b, c are positive numbers such that $abc = 1$, then

$$\frac{a}{ab+a+1} + \frac{b}{bc+b+1} + \frac{c}{ca+c+1} = 1.$$

4.5 Prove that $\sin^2 20° + \sin^2 40° + \sin^2 80° = \frac{3}{2}$.

5.1 Find all integers m and n such that $57m - 87n = 342$.

5.2 $\triangle ABC$ is equilateral and P is a point inside the triangle such that $PQ = 6$, $PR = 8$ and $PS = 10$, as shown. Find the area of $\triangle ABC$.



5.3 Solve: $x^4 - 4x^3 - 2x^2 + 12x + 8 = 0$.

5.4 In a set of 21 numbers, the sum of any 10 is less than the sum of the other 11. Prove that all the numbers are positive.

5.5 Telephone numbers in a certain town have 6 digits. How many telephones can be installed such that any two numbers differ in at least two digits?

6.1 Quadrilateral $ABCD$ is inscribed in a circle centre O, and $AC \perp BD$. Prove that $|AOCB| = |AOCD|$.[1]

6.2 Solve the equation $x! + y! + z! = xyz$, where x, y and z are positive integers and xyz denotes a 3-digit number.

6.3 Solve: $\sqrt{2x^2 + x + 5} + \sqrt{x^2 + x + 1} = \sqrt{x^2 - 3x + 13}$.

6.4 Prove that $2x^4 + 21x^3 - 6x^2 + 9x - 3$ cannot be factorized into two polynomials with integer coefficients.

6.5 In the latest theory of particle physics there are three fundamental particles. When two particles of different types collide they are replaced by a particle of the third type. Two particles of the same type never collide.

Prove that if an experiment begins with equal numbers of particles of each type, it cannot end with just one particle remaining.

[1] Here and subsequently, we use $|\cdot|$ to denote the standard measure of the object indicated. Thus $|x|$ is (of course) the absolute value of x, $|AB|$ is the length of the line segment AB, $|\triangle ABC|$ is the area of the triangle ABC, and $|ABCD|$ is the area of the quadrilateral $ABCD$, etc.

7.1 Find all integers m and n such that $n(n - 2m) = 3m - 7$.

7.2 The 7-digit numbers containing all the digits from 1 to 7 are arranged in ascending order. What is the 1994th number on the list?

7.3 Prove that $\cos 20° \cos 40° \cos 80° = \frac{1}{8}$.

7.4 The four triangular faces of a tetrahedron all have the same perimeter. Prove that they are congruent.

7.5 For which natural numbers n is $n^4 + 4^n$ prime?

2 Stellenbosch University Camp: December 1994

Stellenbosch Camp December 1994: Test I
Time: 2 hours

1. Prove that, if a, b and c are positive numbers such that $a < b + c$, then
$$\frac{a}{1+a} < \frac{b}{1+b} + \frac{c}{1+c}.$$

2. Find a formula for the sum
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}.$$

3. What are the last two digits of 3^{1995}?

4. The number 24 has 8 divisors (1, 2, 3, 4, 6, 8, 12 and 24). Find the smallest number which has 30 divisors.

5. A chord PQ with midpoint R is drawn in a circle with diameter AB. Perpendiculars PS and QT are dropped onto AB.

   (a) Prove that $\triangle RST$ is isosceles.

   (b) Prove that $\triangle RST$ is equilateral if and only if $2PQ = AB$.

6. Which is larger: $\sqrt[3]{60}$, or $2 + \sqrt[3]{7}$?

7. Find all positive integers m and n such that $|2^m - 3^n| = 1$.

Stellenbosch Camp December 1994: Test II
Time: 2 hours

1. Prove that there is precisely one function $f : \mathbb{R} \to \mathbb{R}$ such that

   (a) if $x \leq y$ then $f(x) \leq f(y)$;

   (b) $f(f(x)) = x$ for every x.

2. Determine all positive integers m and n such that $3m + 5n = 1008$.

3. Find all polynomials $p(x)$ with integer coefficients for which we have that $p(0)$, $p(1)$, $\ldots$ are all prime numbers.

4. Find the number of values of x which satisfy the equation
$$3\pi(1 - \cos x) = 2x.$$

5. The rational numbers between 0 and 1 are listed in a sequence
$$0.a_{11}a_{12}a_{13}\ldots,\ 0.a_{21}a_{22}a_{23}\ldots,\ 0.a_{31}a_{32}a_{33}\ldots,\ \ldots$$
Prove that the number $0.a_{11}a_{22}a_{33}\ldots$ is irrational.

Stellenbosch Camp December 1994: Test III
Time: 3 hours

1. Find all solutions of $f(x+y) = y^2 f(x) + y(f(x))^2$.

2. An observer walks along a level road leading directly towards an elevated object P. He takes observations at three points A, B, C, in order, and notes that the angles of elevation of P are θ, 2θ, 3θ. Prove that the ratio $\frac{AB}{BC}$ is greater than 2 and approaches 3 as θ tends to 0.

   (You may assume that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.)

3. Prove that the equation $(x-y)^3 + (y-z)^3 + (z-x)^3 = 30$ has no integer solutions.

4. Find all permutations ϕ such that

$$\phi^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 1 & 6 & 2 & 8 & 5 & 4 & 3 \end{pmatrix}.$$

Stellenbosch Camp December 1994: Test IV
Time: 3 hours

1. A 'prince' is a chesspiece which can move one square to the right or one square upward. In how many ways can a prince move from the square $(1,1)$ to the square (m,n)?

2. A function f defined on the set of integers satisfies

$$f(x) + f(x+3) = x^2$$

   for any integer x. If $f(19) = 94$, find $f(94)$.

3. Let $H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Prove that $H(n)$ is not an integer for all $n > 1$.

4. What algebraic relation holds between α, β and γ if

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\ \tan\beta\ \tan\gamma?$$

Stellenbosch Camp December 1994: Test V
Time: 3 hours

1. Consider products of k different members of $\{1, 2, \ldots, n\}$. How many of these products are divisible by the prime number p?

2. (a) Let $0 < \theta < \frac{\pi}{2}$ and let $t = \tan\frac{\theta}{2}$. Prove that $\sin\theta = \frac{2t}{1+t^2}$ and $\tan\theta = \frac{2t}{1-t^2}$.

   (b) Let P_1 and P_2 be regular polygons of 1995 sides and perimeters of lengths x and y respectively. Each side of P_1 is tangent to a given circle of circumference c and the circle passes through each vertex of P_2. Prove that $x + y \geq 2c$.
   (You may assume that $\tan\theta \geq \theta$ for $0 \leq \theta < \frac{\pi}{2}$.)

3. Let $f : (0,\infty) \to (0,\infty)$ satisfy

   (i) $f(xf(y)) = yf(x)$ for all x and y;

   (ii) $f(x) < 11$ for $x > 1$.

   (a) Prove that $f(f(y)) = y$.

   (b) Prove that if $f(a) = a$ and $f(b) = b$ then $f(ab) = ab$. Deduce that if $f(a) = a$ then $a \leq 1$.

   (c) Now show that $f(x) = \frac{1}{x}$.

4. The sequence (a_n) is formed by the rule

$$a_1 = 1$$
$$a_{n+1} = \begin{cases} 2a_n + 1 & \text{if } n \text{ is odd} \\ 2a_n & \text{if } n \text{ is even} \end{cases}$$

   Find a formula for a_n (as a function of n).

3 The monthly problems: September 1994 to June 1995

1. Find an infinite set A of positive integers with the following property: the sum of the elements of any finite subset of A is not a perfect power.

2. Show that if $x, y, z \in \mathbb{R}$ then

$$|x| + |y| + |z| - |x+y| - |y+z| - |z+x| + |x+y+z| \geq 0.$$

3. Consider the triangle ABC, its circumcircle k of centre O and radius R, and its incircle of centre I and radius r. Another circle k_c is tangent to the sides CA, CB at D, E respectively, and it is internally tangent to k.

   Show that the incentre I is the midpoint of DE.

4. If $\triangle ABC$ has inradius r and circumradius R, show that

$$\cos^2\left(\frac{\angle A - \angle B}{2}\right) \geq \frac{2r}{R}.$$

5. Suppose $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are numbers with the following properties : $0 \leq x_i$, $0 \leq y_i$, $x_i + y_i = 1$ for $i = 1, 2, \ldots, n$. Show that for any $m \in \mathbb{N}$

$$(1 - x_1 \cdot x_2 \cdot \cdots \cdot x_n)^m + (1 - {y_1}^m)\cdot(1 - {y_2}^m)\cdot \cdots \cdot(1 - {y_n}^m) \geq 1.$$

6. Let n be a fixed positive integer. Two players, taking turns, write down positive integers which are less than or equal to n. The rule of the game is that no divisors of numbers already written may appear. The one who cannot move loses. Determine if there is a winning strategy for either player.

7. Determine the maximum value of $m^2 + n^2$, where m and n are integers satisfying $m, n \in \{1, 2, \ldots, 1981\}$ and

$$(n^2 - mn - m^2)^2 = 1.$$

8. Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers (x, y) then it has at least three such solutions. Show that the equation has no solutions in integers when $n = 2891$.

9. Find all functions $f : (0, \infty) \to (0, \infty)$ which satisfy the conditions

   (i) $f(xf(y)) = yf(x)$ for all positive x, y;

   (ii) $f(x) \to 0$ as $x \to \infty$.

10. Six points in space are given so that the pairwise distances between them are all different. Consider the triangles formed by the edges between the points. Prove that the shortest side in one of these triangles is at the same time the longest edge in some other triangle.

11. Suppose $\alpha + \beta + \gamma = \pi$ (i.e. $180°$). Show that

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4 \sin\alpha \sin\beta \sin\gamma.$$

12. ΔABC has $\angle C = 90°$, and is inscribed inside a circle. Let H be the foot of the altitude from C and M the midpoint of BC. A second circle passes through A and M and is tangent to the first circle. This circle intersects BC at N as well as M. Prove that the line AN passes through the midpoint of CH.

13. Solve the following system of equations, in which $a \in \mathbb{R}$ satisfies $|a| > 1$:

$$\begin{aligned} x_1^2 &= ax_2 + 1 \\ x_2^2 &= ax_3 + 1 \\ \vdots \;\; & \;\; \vdots \;\; \vdots \\ x_{999}^2 &= ax_{1000} + 1 \\ x_{1000}^2 &= ax_1 + 1. \end{aligned}$$

14. Given a plane π, a point P in this plane and a point Q not in π, find all points R in π such that the ratio $\dfrac{QP + PR}{QR}$ is a maximum.

15. Suppose $\alpha + \beta + \gamma = \pi$. Show that

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -4\cos\alpha\cos\beta\cos\gamma - 1.$$

16. Suppose $f(x)$ is a positive valued concave (i.e. the graph lies above the chord) function defined on some interval. Show that $\ln(f(x))$ is also a concave function.

17. Suppose a polygon is inscribed in a circle of radius one. Determine the maximum possible value of the sum of the squares of the lengths of the sides of the polygon.

18. Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incentre and the circumcentre of the triangle and the point O are collinear.

19. Three roots of the equation $x^4 - px^3 + qx^2 - rx + s = 0$ are $\tan A$, $\tan B$, $\tan C$ where A, B, C are the angles of a triangle. Determine the fourth root as a function of p, q, r, s.

20. (a) For which values of $n > 2$ is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining $n - 1$ numbers?

    (b) For which values of $n > 2$ is there exactly one set having the stated property?

21. Suppose $0 \le x_1, x_2, \ldots, x_n$ with $n \ge 4$, and $x_1 + x_2 + \cdots + x_n = 1$. Show that

$$x_1x_2 + x_2x_3 + \cdots + x_nx_1 \le \frac{1}{4}.$$

22. Let $a_1, a_2, \ldots, a_n$ be arbitrary real numbers. Prove that there exists a real number x such that each of $x + a_1, x + a_2, \ldots, x + a_n$ is irrational.

23. Find all continuous functions $g : \mathbb{R} \to \mathbb{R}$ which satisfy

$$g(x + y) + g(x - y) = 2g(x) + 2g(y).$$

24. A non-isosceles triangle $A_1A_2A_3$ is given. For $i = 1, 2, 3$, M_i is the midpoint of side a_i and T_i is the point where the incircle touches side a_i. Denote by S_i the reflection of T_i in the interior bisector of angle A_i. Prove that the lines M_1S_1, M_2S_2 and M_3S_3 are concurrent.

25. Define a function $f : \mathbb{N} \to \mathbb{N}$ iteratively by:-

- $f(1) = 1$;
- For $n \geq 2$, $f(n) = \begin{cases} f(n-1) - n & \text{if} \quad f(n-1) > n \\ f(n-1) + n & \text{if} \quad f(n-1) \leq n \end{cases}$

Let $S = \{n \in \mathbb{N} : f(n) = 1993\}$.

(i) Prove that S is an infinite set.

(ii) Find the least member of S.

(iii) If all the members of S are written in ascending order as $n_1 < n_2 < \cdots$, show that $\lim_{i\to\infty} \frac{n_{i+1}}{n_i} = 3$.

26. A permutation $(x_1, \ldots, x_{2n})$ of the set $\{1, 2, \ldots, 2n\}$ is said to have property T if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \ldots, 2n-1\}$.

Show that, for each n, there are more permutations with property T than without.

27. Two pyramids with common base $A_1A_2A_3A_4A_5A_6A_7$ and vertices B and C are given. The edge BC, the edges BA_i and CA_i, and the diagonals of the base are coloured either red or blue. Show that there is a monochromatic triangle.

28. An arbitrary natural number k is given. Prove that there exists a prime p and a strictly increasing positive integer sequence $a_1, a_2, \ldots$ such that the terms of the sequence $p+ka_1$, $p+ka_2$, $\ldots$ are all primes.

29. Suppose ABC is a triangle, and P is some interior point. Let AP, BP, CP meet BC, CA, AB at D, E, F. Find for which location of P the area of DEF will be maximized.

30. (a) Find a sequence $a_0, a_1, a_2, \ldots$ with the following properties:-

- $a_0 = 1$;
- $a_i \geq 0$ for all $i \in \mathbb{N}$;
- $a_{n+2} = a_n - a_{n+1}$.

(b) Show that the sequence you have found is uniquely determined.

31. Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n + 1987$ for every n.

32. To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all the five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively and $y < 0$ then the following operation is allowed: the numbers x, y, z are replaced by $x+y$, $-y$, $z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

33. Let $a_1, a_2, \ldots, a_{10}$ be the various possible pairwise sums of the numbers $x_1, x_2, \ldots, x_5$.

Given $a_1, a_2, \ldots, a_{10}$, find $x_1, x_2, \ldots, x_5$.

34. Let n be an integer greater than 1 and let $a_1, a_2, \ldots, a_n$ be n distinct integers. Prove that the polynomial $f(x) = (x-a_1)(x-a_2)\cdots(x-a_n) - 1$ is irreducible in $\mathbb{Q}[x]$.

35. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that, for some positive constant a, we have

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - (f(x))^2}$$

(i) Prove that f is periodic, that is, there exists some positive b such that $f(x+b) = f(x)$ for all $x \in \mathbb{R}$.

(ii) With $a = 1$, give an example of a non-constant function which satisfies the given functional equation.

36. For a given prime p, find the greatest positive integer n with the following property: the edges of the complete graph with n vertices can be coloured in $p+1$ colours so that:

- at least two different edges have different colour;
- there are no dichromatic triangles (i.e. all triangle are trichromatic or monochromatic).

37. Show that the set of real numbers x which satisfy the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths is 1988.

38. Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centres O_1 and O_2. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1 and M_2 the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.

4 Rhodes University Camp: April 1995

Rhodes Camp April 1995: Test I
Time: 3 hours

1. Two chords AB, CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent line at E to the circle through D, E, M intersects the lines BC, AC at F, G respectively. If $\frac{AM}{AB} = t$, find $\frac{EG}{EF}$ in terms of t.

2. Find all integers a, b, c with $1 < a < b < c$ such that $(a-1)(b-1)(c-1)$ is a divisor of $abc - 1$.

Rhodes Camp April 1995: Test II
Time: 3 hours

1. A circle has its centre O on the side AB of a cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that

$$AD + BC = AB.$$

2. In a given plane, let K and k be circles with radii R and r respectively, and suppose that K and k intersect in precisely two points S and T. Let the tangent to k through S intersect K also in B, and suppose that B lies on the common tangent to k and K.

   Prove that if ϕ is the interior angle between the tangents of K and k at S, then

$$\frac{r}{R} = \left(2\sin\frac{\phi}{2}\right)^2.$$

Rhodes Camp April 1995: Test III
Time: 3 hours

1. Prove that for any positive integer m there exists an infinite number of pairs of integers (x, y) such that
   - x and y are relatively prime;
   - y divides $x^2 + m$;
   - x divides $y^2 + m$.

2. Given a triangle ABC, let I be the centre of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that

$$\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}.$$

Rhodes Camp April 1995: Test IV
Time: 4.5 hours

1. Let $a_1, a_2, \ldots$ be an infinite increasing sequence of positive integers. Prove that for every $p \geq 1$ there are infinitely many a_m which can be written in the form

$$a_m = xa_p + ya_q$$

for some natural numbers x, y and some $q > p$.

2. Suppose $ABCDEF$ is a regular hexagon. Let M be on the line segment AC and N be on the line segment CE with $AM = CN$. Furthermore suppose that B, M, N are collinear. Determine the ratio $AC : AM$.

3. For any positive integer x define

   - $g(x) =$ greatest odd divisor of x;
   - $f(x) = \begin{cases} \frac{x}{2} + \frac{x}{g(x)} & \text{if } x \text{ is even} \\ 2^{\frac{x+1}{2}} & \text{if } x \text{ is odd.} \end{cases}$

   Construct the sequence $x_1 = 1$, $x_{n+1} = f(x_n)$. Show that the number 1992 appears in the sequence, determine the least n such that $x_n = 1992$, and find out whether n is unique.

Rhodes Camp April 1995: Test V
Time: 4.5 hours

1. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges the greatest common divisor of the integers labelling those edges is equal to 1.

2. ABC is a triangle right-angled at A, and D is the foot of the altitude from A. The straight line joining the incentres of the triangles ABD, ACD intersects the sides AB, AC at the points K, L respectively. S and T denote the areas of the triangles ABC and AKL respectively. Show that $S \geq 2T$.

3. Is it possible to choose 1983 distinct positive integers, all less than or equal to 100000, no three of which are in arithmetic progression?

The South African team was announced at the end of the Rhodes University camp.

Mark Berman (standard 10, Diocesan College)
David Fraser (standard 10, Westerford High School)
David Hatton (standard 9, Welkom High School)
Elitca Mitova (standard 10, Durban Girls' High School)
Andrew Skeen (standard 10, St John's College)
Jan van Zyl Smit (standard 9, Diocesan College)
Tim Lawrance (standard 9, Hilton College) - reserve.

Prof John Webb was the team leader and Dr Graeme West the deputy team leader.

5 Wits University Camp: June/July 1995

Wits Camp June/July 1995: Test I
Time: 4.5 hours

1. Consider two coplanar circles of radii R and r ($r < R$) with the same centre. Let P be a fixed point on the smaller circle and B a variable point on the larger circle. The line BP meets the larger circle again at C. The perpendicular ℓ to BP at P meets the smaller circle again at A (if ℓ is tangent to the circle at P then $A = P$).

   (i) Find the set of values of $BC^2 + CA^2 + AB^2$.

   (ii) Find the locus of the midpoint of AB.

2. Find one pair of positive integers a and b such that:

   (i) $ab(a+b)$ is not divisible by 7;

   (ii) $(a+b)^7 - a^7 - b^7$ is divisible by 7^7.

3. Let S be a square with sides of length 100, and let L be a path within S which does not meet itself and which is composed of line segments A_0A_1, A_1A_2, $\ldots$, $A_{n-1}A_n$ with $A_0 \neq A_n$. Suppose that for every point P on the boundary of S there is a point of L at a distance from P not greater than $1/2$. Prove that there are two points X and Y in L such that the distance between X and Y is not greater than 1, and the length of that part of L which lies between X and Y is not smaller than 198.

Wits Camp June/July 1995: Test II
Time: 4.5 hours

1. In $\triangle ABC$, with $\angle A = 60^\circ$, a parallel IF to AC is drawn through the incentre I of the triangle, where F lies on the side AB. The point P on the side BC is such that $3\,BP = BC$. Show that $\angle BFP = \frac{1}{2}\angle B$.

2. Let $\triangle ABC$ be equilateral and $\mathcal{S}$ the set of all points contained in the three segments AB, BC, CA (including A, B, C). Determine whether, for every partition of $\mathcal{S}$ into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle.

3. Consider the infinite sequences (x_n) of positive real numbers with $x_0 = 1$ and $x_{i+1} \leq x_i$ for all $i \geq 0$.

   (a) Prove that for every such sequence, there is an $n \geq 1$ such that
   $$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

   (b) Find such a sequence for which
   $$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} < 4$$
   for all $n \in \mathbb{N}$.

Wits Camp June/July 1995: Test III
Time: 4.5 hours

1. Determine the maximum value of the sum
   $$\sum_{i<j} x_i x_j (x_i + x_j)$$
   over all n-tuples $(x_1, x_2, \ldots, x_n)$ where $x_i \geq 0$, $\sum_{i=1}^n x_i = 1$.

2. Let A, B be adjacent vertices of a regular n-gon $(n \geq 5)$ in the plane having centre at O. A triangle XYZ, which is congruent to and initially coincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, X remaining inside the polygon. Find the locus of X.

3. Let a, b, c, d be odd integers with $0 < a < b < c < d$ and $ad = bc$. Prove that if both $a + d$ and $b + c$ are powers of 2, then $a = 1$.

Wits Camp June/July 1995: Test IV
Time: 4.5 hours

1. In a triangle ABC, let K, L, M be arbitrary points on BC, CA, AB and let N, R, F be arbitrary points on LM, MK, KL. If E_1, E_2, …, E_6 and E denote the respective areas of the triangles AMR, CKR, BKF, ALF, BNM, CLN, ABC, show that

$$E \geq 8(E_1E_2E_3E_4E_5E_6)^{1/6}.$$

2. Given a set $\mathcal{M}$ of 1985 distinct positive integers, none of which has a prime divisor greater than 26. Prove that $\mathcal{M}$ contains at least one subset of four distinct elements whose product is the fourth power of an integer.

3. On a 5×5 board, two players alternatively mark numbers in empty cells. The first player always marks 1's, the second 0's. One number is marked per turn, until the board is filled. For each of the 3×3 squares on the board, the sum of the nine entries is calculated. How large can the first player make the maximum of these sums, regardless of the responses of the second player?

Wits Camp June/July 1995: Test V
Time: 4.5 hours

1. Suppose 7 points in the plane are given. What is the fewest number of edges required so that at least two of every three points are joined by an edge?

2. Let n and k be given relatively prime natural numbers, with $k < n$. Each number in the set $M = \{1, 2, \ldots, n-1\}$ is coloured either red or blue. It is given that

   (i) for each $i \in M$, both i and $n - i$ have the same colour;

   (ii) for each $i \in M$, $i \neq k$, both i and $|i - k|$ have the same colour.

   Prove that all the numbers in M must have the same colour.

3. In the plane two different points O and A are given. For each point X of the plane, other than O, denote by $a(X)$ the measure of the angle between OA and OX in radians, counterclockwise from OA $(0 \leq a(X) < 2\pi)$. Let $c(X)$ be the circle with centre O and radius of length $OX + \frac{a(X)}{OX}$. Each point of the plane is coloured by one of a finite number of colours. Prove that there exists a point Y for which $a(Y) > 0$ such that its colour appears on the circumference of the circle $c(Y)$.

6 Kent State University Camp: July 1995

Kent Camp July 1995: Test I
Time: 4.5 hours

1. Prove that the set $\{1, 2, \ldots, 1989\}$ can be expressed as the disjoint union of subsets A_i ($i = 1, 2, \ldots, 117$) such that

   (i) each A_i contains 17 elements;

   (ii) the sum of all of the elements in each A_i is the same.

2. In $\triangle ABC$, let D and E be the intersections of the bisectors of $\angle ABC$ and $\angle ACB$ with the sides AC and AB respectively. Determine the angles $\angle A$, $\angle B$, $\angle C$ if $\angle BDE = 24°$ and $\angle CED = 18°$.

3. One is given a finite set of points in the plane, each point having integer coordinates. Is it always possible to colour some of the points in the set red and the remaining points blue in such a way that for any straight line L parallel to either one of the coordinate axes the difference (in absolute value) between the numbers of blue points and red points on L is not greater than 1? Justify your answer.

Kent Camp July 1995: Test II
Time: 4.5 hours

1. Let $ABCD$ be a convex quadrilateral such that the line CD is a tangent to the circle that has AB as a diameter. Prove that the line AB is a tangent to the circle which has CD as a diameter if and only if the lines BC and AD are parallel.

2. Let $n \geq 3$ and consider a set $\mathcal{E}$ of $2n-1$ distinct points on a circle. Suppose that exactly k of these points are to be coloured black. Such a colouring is good if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from $\mathcal{E}$. Find the smallest value of k so that every such colouring of k points of $\mathcal{E}$ is good.

3. At a round table are 1994 children, playing a game with a deck of n cards. Initially, one child holds all the cards. In each turn, if at least one child holds at least two cards, one of these children must pass a card to each of his/her neighbours. The game ends if and only if each child is holding at most one card.

   Prove that the game ends if and only if $n < 1994$.

Kent Camp July 1995: Test III
Time: 4.5 hours

1. Let $p_n(k)$ be the number of permutations of $\{1, 2, \ldots, n\}$ which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k \cdot p_n(k) = n!$$

2. Suppose $g : \mathbb{C} \to \mathbb{C}$ and a, $w \in \mathbb{C}$ are fixed, with $w^3 = 1$ but $w \neq 1$. Find all functions $f : \mathbb{C} \to \mathbb{C}$ that satisfy

$$f(z) + f(wz + a) = g(z).$$

3. Consider a segment of a circle which is cut off by a chord BC. Suppose two circles are inscribed in the segment so that they touch each other externally at a point W. Let A be the point of intersection of the common internal tangent of the two inscribed circles with the arc of the segment.

   Show that W is the incentre of $\triangle ABC$.

Kent Camp July 1995: Test IV
Time: 4.5 hours

1. Let $f(x) \in \mathbb{Z}[x]$ be monic of degree 1995. Let $g(x) = f(x)^2 - 9$. Show that the number of distinct integer solutions of $g(x)$ cannot exceed 1995.

2. A circle with centre O passes through the vertices A and C of $\triangle ABC$ and intersects the segments AB and BC again at distinct points K and N, respectively. The circumscribed circles of $\triangle ABC$ and $\triangle KBN$ intersect at exactly two distinct points B and M. Prove that $\angle OMB$ is a right angle.

3. There are $n+1$ fixed positions in a row, labelled 0 to n in increasing order from right to left. Cards numbered 0 to n are shuffled and dealt, one to each position. A game is played, the object of which is to have the card i in the i^{th} position for $0 \leq i \leq n$. If this has not yet been achieved, the following move is executed: determine the smallest k such that the k^{th} position is occupied by a card $l > k$, pick up this card, slide all cards from the $k+1^{st}$ to the l^{th} position one place to the right, and replace card l in the l^{th} position.

   (a) Prove that the game lasts at most $2^n - 1$ moves.

   (b) Prove that there exists at most one initial configuration for which the game lasts exactly $2^n - 1$ moves.

   (c) Give the initial configuration from (b).

Kent Camp July 1995: Test V
Time: 3 hours

1. Show that $\dfrac{5^{125}-1}{5^{25}-1}$ is composite.

2. Let a be a rational number with $0 < a < 1$ and suppose that

$$\cos(3\pi a) + 2\cos(2\pi a) = 0.$$

(Angle measurements are in radians.) Prove that $a = 2/3$.

7 The IMO in Toronto: July 1995

Each paper at the IMO consists of three problems, to be solved in 4 1/2 hours. The papers are written on successive days. Each question is worth 7 points, with a maximum score of 42.

IMO 1995: Test I
Time: 4.5 hours

1. Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN and XY are concurrent.

2. Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

3. Determine all integers $n > 3$ for which there exist n points in the plane $A_1, A_2, \ldots, A_n$, and n real numbers $r_1, r_2, \ldots, r_n$, satisfying the following two conditions:

   (i) no three of the points $A_1, A_2, \ldots, A_n$ lie on a line;

   (ii) for each triple i, j, k ($1 \leq i < j < k \leq n$) the triangle $A_iA_jA_k$ has area equal to $r_i + r_j + r_k$.

CANADA
1995

IMO 1995: Test II
Time: 4.5 hours

4. Find the maximum value of x_0 for which there exists a sequence of positive real numbers $x_0, x_1, \ldots, x_{1995}$ satisfying the two conditions:

   (i) $x_0 = x_{1995}$

   (ii) $x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}$ for each $i = 1, 2, \ldots, 1995$.

5. Let ABCDEF be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, and $\angle BCD = \angle EFA = 60°$. Let G and H be two points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120°$. Prove that

$$AG + GB + GH + DH + HE \geq CF.$$

6. Let p be an odd prime number. Find the number of subsets of A of the set $\{1, 2, \ldots, 2p\}$ such that

   (i) A has exactly p elements, and

   (ii) the sum of all the elements in A is divisible by p.

8 Solutions to the problems

8.1 Solutions to the Talent Search

1.1

$$\begin{aligned}
& (a-b)^2 + (b-c)^2 + (c-a)^2 \\
= \; & a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 - 2ca + a^2 \\
= \; & 3(a^2 + b^2 + c^2) - (a^2 + b^2 + c^2 + 2ab + 2bc + 2ca) \\
= \; & 3 - (a + b + c)^2 \\
\leq \; & 3
\end{aligned}$$

1.2 Suppose that $2(m^2+mn+n^2)$ is a perfect square. Then, since even squares are multiples of 4, m^2+mn+n^2 is even. It follows that both m and n are even. (They cannot both be odd, for then m^2+mn+n^2 is odd, and if one is odd and the other even, m^2+mn+n^2 is again odd.) So $m = 2M$, $n = 2N$ and $2((2M)^2 + (2M)(2N) + (2N)^2) = 8(M^2 + MN + N^2)$ is a perfect square. Hence $2(M^2 + MN + N^2)$ is a perfect square. Repeating the argument, M and N are both even, and we can get a smaller pair of integers (P, Q) with $2(P^2 + PQ + Q^2)$ a perfect square. This process can be continued indefinitely, which is a contradiction.

We conclude that no integers m, n can be found such that $2(m^2 + mn+n^2)$ is a perfect square. The method of proof here is important. It is known as the Principle of Infinite Descent, and is logically equivalent to the Principle of Mathematical Induction.

1.3 By Menelaus' Theorem[2] $\frac{AO}{OD} \cdot \frac{DB}{BC} \cdot \frac{CE}{EA} = 1$. Since $\frac{DB}{BC} = \frac{4}{11}$ and $\frac{CE}{EA} = \frac{3}{2}$, $\frac{AO}{OD} = \frac{11}{6}$.

1.4 A regular m-gon inscribed in a circle of radius r is made up of m isosceles triangles of area $\frac{1}{2}r^2 \sin\left(\frac{360°}{m}\right)$. From

$$\frac{m}{n} = \frac{|m\text{-gon}|}{|n\text{-gon}|} = \frac{\frac{1}{2}r^2 \sin\left(\frac{360°}{m}\right) \cdot m}{\frac{1}{2}r^2 \sin\left(\frac{360°}{n}\right) \cdot n}$$

[2] see, for example, *Mathematical Digest* October 1993, p. 12

we deduce that $\sin\left(\frac{360°}{m}\right) = \sin\left(\frac{360°}{n}\right)$ and hence either $m = n$ or $\frac{360°}{m} = 180° - \frac{360°}{n}$. In the second instance we have $\frac{2}{m} = 1 - \frac{2}{n}$, which quickly reduces to $(m-2)(n-2) = 4$. This has the solutions $m = 6$, $n = 3$ and $m = 3$, $n = 6$. (The solution $m = n = 4$ has already appeared.)

1.5 Experiment shows that 38 is not expressible as the sum of two odd composite numbers. Even numbers from 40 onwards can be put into five classes:

$$\begin{aligned} 40 + 10k &= 25 + 5(2k+3) \\ 42 + 10k &= 27 + 5(2k+3) \\ 44 + 10k &= 9 + 5(2k+7) \\ 46 + 10k &= 21 + 5(2k+5) \\ 48 + 10k &= 33 + 5(2k+3) \end{aligned}$$

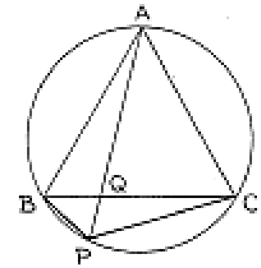
for $k = 0, 1, 2, \ldots$ and are thus expressible as the sum of two composite odd numbers.

2.1 Trial and error shows that $36 = 3 \cdot 7 + 3 \cdot 5$, $37 = 1 \cdot 7 + 6 \cdot 5$, $38 = 4 \cdot 7 + 2 \cdot 5$, $39 = 2 \cdot 7 + 5 \cdot 5$ and $40 = 5 \cdot 7 + 1 \cdot 5$, and higher numbers may be obtained by adding multiples of 5 to these sums. On the other hand, 35 cannot be thus obtained. For if $35 = 5x + 7y$, then $5x \equiv_7 0$, and so $x \equiv_7 0$, since 5 and 7 are relatively prime. Thus $x = 0$, a contradiction.

2.2 Since $\angle BPQ = \angle QPC = 60°$, the equation

$$|\triangle PBQ| + |\triangle PQC| = |\triangle BPC|$$

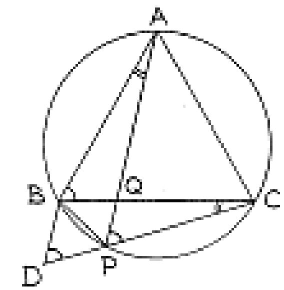
becomes $\frac{1}{2}BP \cdot PQ \cdot \sin 60° + \frac{1}{2}PQ \cdot PC \cdot \sin 60° = \frac{1}{2}BP \cdot PC \cdot \sin 120°$. Dividing by $\frac{1}{2}BP \cdot PQ \cdot PC \cdot \sin 60°$ gives $\frac{1}{PC} + \frac{1}{PB} = \frac{1}{PQ}$.



Alternatively, produce CP to D with $PD = PB$.
$\angle BPD = \angle BAC = 60°$, so $\triangle BPD$ is equilateral. So $\triangle DCB \,|||\, \triangle PCQ$. Hence

$$\frac{DB}{PQ} = \frac{DC}{PC} = \frac{DP + PC}{PC} = \frac{DP}{PC} + 1.$$

So $\frac{PB}{PQ} = 1 + \frac{BP}{PC}$ since $DB = PB$, and the result follows.



2.3 Put $a = \sqrt{2x-1}$ and $b = \sqrt{y+3}$. Then $a + b = 3$ and, since $(2x-1)(y+3) = 2xy + 6x - y - 3$, $a^2b^2 = 4$. Thus $ab = 2$, since a and b are non-negative. Solving $a + b = 3$ and $ab = 2$ gives $a = 2$, $b = 1$ (and so $x = \frac{5}{2}$, $y = -2$) or $a = 1$, $b = 2$ (and so $x = y = 1$).

2.4 Let a_n denote the number of n-digit strings without consecutive zeros. There are two possibilities:

(i) The first digit of the string is zero. Then its second digit must be non-zero (9 possibilities) and there are a_{n-2} possibilities for the remaining $n-2$ digits. So there are $9a_{n-2}$ n-digit strings beginning with 0, without any consecutive zeros.

(ii) The first digit of the string is non-zero. There are then a_{n-1} possibilities for the remaining $n-1$ digits. So there are $9a_{n-1}$ n-digit strings, with first digit non-zero, and no consecutive zeros.

We thus deduce that $a_n = 9a_{n-2} + 9a_{n-1}$.

Now $a_1 = 10$ and $a_2 = 99$. So $a_3 = 981$, $a_4 = 9720$, $a_5 = 96309$.

2.5 Since $4^n + 2$ is even for every positive integer n, we need only show that $4^n + 2$ is divisible by 3, which is equivalent to proving that $4^n - 1$ is divisible by 3. But by the remainder theorem (or indeed, by the formula for a geometric progression) we have that $x^n - 1$ is divisible by $x - 1$. Putting $x = 4$, we get what we want.

3.1 At x minutes past 5 the minute hand is at $(6x)^\circ$ to the vertical, while the hour hand is at $150^\circ + (x/2)^\circ$ to the vertical. So if the hands are at 90° we have that $150 + \frac{x}{2} - 6x = \pm 90$. The solutions to this equation are $x = \frac{120}{11}$ and $x = \frac{480}{11}$. So the time between the two events is $\frac{480}{11} - \frac{120}{11} = \frac{360}{11} = 32\frac{8}{11}$ minutes.

3.2

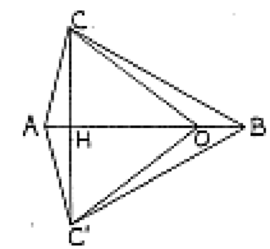
$$\left(7^{\sqrt{5}}\right)^{\sqrt{5}} = 7^5 = 16807 > 15625 = 5^6 = 5^{\sqrt{36}} > 5^{\sqrt{35}} = \left(5^{\sqrt{7}}\right)^{\sqrt{5}}$$

and so $7^{\sqrt{5}} > 5^{\sqrt{7}}$.

3.3 Put $x = y = 0$, getting $f(0)^2 - f(0) = 0$. Thus $f(0) \in \{0, 1\}$. But putting $x = 0$ and $y = 1$ gives $f(0)f(1) - f(0) = 1$, and so $f(0) \neq 0$. Thus $f(0) = 1$. Now put $y = 0$, getting $f(x)f(0) - f(0) = x + 0$, that is, $f(x) = x + 1$.

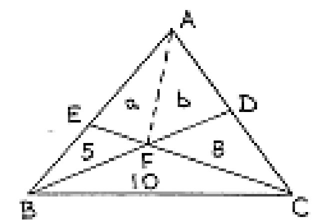
3.4 If any two numbers differ by a multiple of 9, we are done. If not, then all six numbers leave different remainders on division by 9. Now consider the five pigeonholes $\{0\}, \{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}$. Assign the six integers to the pigeonholes. By the pigeonhole principle two integers go into the same pigeonhole. Since they have different remainders on division by 9, their sum will be divisible by 9.

3.5 Produce CH to C' in such a way that $HC' = CH$. Then $CC' = 2CH = AB$.
Denote by O the circumcentre of $\Delta ACC'$. Then O lies on AB.
Reflex $\angle COC' = 2 \cdot 150^\circ = 300^\circ$, so $\angle COC' = 60^\circ$ and hence $\Delta COC'$ is equilateral.



Since $\angle ACH = 15^\circ$, $\angle ACO = 75^\circ$ and hence $OA = OC$. But then $AB = CC' = CO = AO$. So $B = O$. Hence $\angle ABC = 30^\circ$.

4.1 We have



$$\frac{|AEC|}{|AEF|} = \frac{|EBC|}{|EBF|}$$

and

$$\frac{|ADB|}{|ADF|} = \frac{|DCB|}{|DCF|}$$

giving $\frac{a+b+8}{a} = \frac{15}{5} = 3$, so $-2a + b + 8 = 0$; and $\frac{a+b+5}{b} = \frac{18}{8}$, so $4a - 5b + 20 = 0$. Thus $a = 10$, $b = 12$ and $x = a + b = 22$.

4.2 Using the factorisation

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

we have

$$\begin{aligned} m^3 + (-n)^3 + (-2)^3 - 3m(-n)(-2) &= 0 \\ \Leftrightarrow (m - n - 2)(m^2 + n^2 + 4 + 2m - 2n + mn) &= 0 \\ \Leftrightarrow (m - n - 2)\left[\frac{1}{2}(m + n)^2 + \frac{1}{2}(m + 2)^2 + \frac{1}{2}(n - 2)^2\right] &= 0 \end{aligned}$$

and so $m - n - 2 = 0$ or $m = -2$, $n = 2$. So the solution set is $\{(n + 2, n) : n \in \mathbb{Z}\} \cup \{(-2, 2)\}$.

4.3 Put a, b, c into two pigeonholes: numbers congruent to 1 or 4 modulo 5, and numbers congruent to 2 or 3 modulo 5. Two numbers (say a and b) are in the same pigeonhole, and their sum is then divisible by 5. Hence $a^2 - b^2 = (a - b)(a + b)$ is divisible by 5.

4.4

$$
\begin{aligned}
& \frac{a}{ab+a+1}+\frac{b}{bc+b+1}+\frac{c}{ca+c+1} \\
= & \frac{a}{ab+a+1}+\frac{ab}{abc+ab+a}+\frac{abc}{abca+abc+ab} \\
= & \frac{a}{ab+a+1}+\frac{ab}{1+ab+a}+\frac{1}{a+1+ab} \\
= & 1
\end{aligned}
$$

4.5

$$
\begin{aligned}
& \sin^2 20^\circ+\sin^2 40^\circ+\sin^2 80^\circ \\
= & \sin^2 20^\circ+\sin^2(60^\circ-40^\circ)+\sin^2(60^\circ+20^\circ) \\
= & \sin^2 20^\circ+[\sin 60^\circ\cos 20^\circ-\cos 60^\circ\sin 20^\circ]^2 \\
= & \quad +[\sin 60^\circ\cos 20^\circ+\cos 60^\circ\sin 20^\circ]^2 \\
= & \sin^2 20^\circ+2\sin^2 60^\circ\cos^2 20^\circ+2\cos^2 60^\circ\sin^2 20^\circ \\
= & \sin^2 20^\circ+\frac{3}{2}(1-\sin^2 20^\circ)+\frac{1}{2}\sin^2 20^\circ \\
= & \frac{3}{2}
\end{aligned}
$$

5.1 $57m-87n=342$ implies that $19m-29n=114=6\cdot 19$, and so n is a multiple of 19. Put $n=19k$. Then $m=6+29k$.

5.2 $|ABC|=|PAB|+|PBC|+|PCA|$, or $\frac{\sqrt{3}}{4}a^2=\frac{1}{2}\cdot 10a+\frac{1}{2}\cdot 6a+\frac{1}{2}\cdot 8a$. Thus $a=16\sqrt{3}$, and so $|ABC|=192\sqrt{3}$.

5.3

$$
\begin{aligned}
x^4-4x^3 &= 2x^2-12x-8 \\
\Leftrightarrow\ x^4-4x^3+6x^2-4x+1 &= 2x^2-12x-8+6x^2-4x+1 \\
\Leftrightarrow\ (x-1)^4 &= 8x^2-16x-7 \\
\Leftrightarrow\ (x-1)^4-8(x-1)^2-15 &= 0 \\
\Leftrightarrow\ [(x-1)^2-3][(x-1)^2-5] &= 0 \\
\Leftrightarrow\ x &\in \{1\pm\sqrt{3},\ 1\pm\sqrt{5}\}
\end{aligned}
$$

5.4 Denote the numbers by a_n $(1\le n\le 21)$ and suppose that $a_1\le a_2\le\cdots\le a_{21}$. Then

$$
\begin{aligned}
a_1+a_2+\cdots+a_{11} &> a_{12}+a_{13}+\cdots+a_{21} \\
&\ge a_2+a_3+\cdots+a_{11}
\end{aligned}
$$

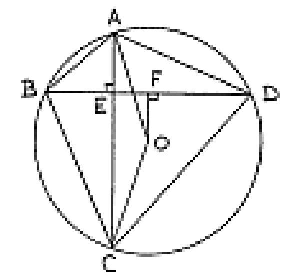
Hence $a_1>0$, and all the numbers are positive.

5.5 We prove first that the maximum size is 10^5 numbers. For among 10^5+1 six-digit numbers, there will be at least 10^4+1 with the same first digit. Of these, there will be at least 10^3+1 with the same second digit (hence having the same first two digits). Continuing in this way, there will be at least 10^2+1 numbers with the same first three digits, 11 with the same first four digits, and 2 with the same first five digits, that is, differing only in the sixth digit.

So the largest set of telephone numbers will have no more than 10^5 numbers. Such a set can be constructed by choosing the first five digits arbitrarily (10^5 choices) and then choosing the last digit to make the sum of the digits divisible by 10. Two numbers in this set will differ at least once in their first five digits. If they differ exactly once in their first five digits, their last digit will be different. So any two numbers will differ in at least two digits.

6.1

$$
\begin{aligned}
& |AOCB| \\
= & |ABC|+|AOC| \\
= & \frac{1}{2}AC\cdot BE+\frac{1}{2}AC\cdot EF \\
= & \frac{1}{2}AC\cdot BF \\
= & \frac{1}{4}AC\cdot BD \\
= & \frac{1}{4}AC(BE+ED) \\
= & \frac{1}{2}(|ABC|+|ACD|) \\
= & \frac{1}{2}|ABCD|.
\end{aligned}
$$



Hence $|AOCD| = |ABCO|$.

6.2 A little experimentation gives the solution $1! + 4! + 5! = 145$. We show that this is the only solution. We have: $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$, $7! = 5040$. So none of the digits is 7 or more, and in fact none of the digits can be 6. On the other hand, at least one of x, y, z is 5. All three are not 5, since $5!+5!+5! \neq 555$. If two were 5, they would have to be y and z, and then $x = 2$. But $2! + 5! + 5! \neq 255$. So only one is 5, and it is not x. In fact, $x = 1$. Now trial and error shows that the above solution is unique.

6.3

$$\begin{aligned}
\sqrt{2x^2+x+5} + \sqrt{x^2+x+1} &= \sqrt{x^2-3x+13} \\
\Leftrightarrow 3x^2+2x+6+2\sqrt{(2x^2+x+5)(x^2+x+1)} &= x^2-3x+15 \\
\Leftrightarrow 2\sqrt{(2x^2+x+5)(x^2+x+1)} &= -2x^2-5x+7.
\end{aligned}$$

Since the three quadratics are all positive, squaring introduces no extra solutions.

At this stage squaring again is not promising, as it will produce a quartic which may have extraneous roots. However, the inspired observation that $-2x^2-5x+7 = 3(2x^2+x+5)-8(x^2+x+1)$ reduces the equation to $2\sqrt{uv} = 3u-8v$ where $u = 2x^2+x+5$ and $v = x^2+x+1$. The equation now factorises as $(3\sqrt{u}+4\sqrt{v})(\sqrt{u}-2\sqrt{v}) = 0$ and since the first factor is positive we now have $\sqrt{u}-2\sqrt{v} = 0$. This quickly produces $x = \frac{-3\pm\sqrt{17}}{4}$.

6.4 This is an immediate consequence of Eisenstein's irreducibility criterion. [3]

[3] Eisenstein's irreducibility criterion says that if $f(x) = a_0 + a_1x + \cdots a_nx^n \in \mathbb{Z}[x]$ and there exists some prime number p such that $p^2 \nmid a_0$, $p|a_0$, $p|a_1$, $\ldots$, $p|a_{n-1}$ and $p \nmid a_n$, then $f(x)$ is irreducible in $\mathbb{Z}[x]$. The proof goes as follows: suppose on the contrary that $f(x) = g(x)\cdot h(x)$ is a non-trivial factorisation, and let's put $g(x) = b_0 + b_1x + \cdots + b_kx^k$ and $h(x) = c_0 + c_1x + \cdots + c_mx^m$. Thus $0 < k$, $m < n$. Since $a_0 = b_0c_0$, we have that (say) $b_0 \equiv_p 0$ but $c_0 \not\equiv_p 0$. Let r be the greatest index such that $b_r \equiv_p 0$. Now $a_r = b_rc_0 + b_{r-1}c_1 + \cdots + b_0c_r \equiv_p b_rc_0 \not\equiv_p 0$, and so $r = n$. Thus we have a contradiction: $n = r \leq k < n$.

An ad hoc argument in the special case posed is also possible.

6.5 Suppose there are n particles of each type initially. At each collision, the number of particles of one type increases by one, and the number of particles of the other two types each decrease by one. Initially the numbers all have the same parity, and after each collision the numbers all change parity. So a final situation of just one particle remaining is impossible, since the numbers 0, 0, 1 do not have the same parity.

7.1 $m = \frac{n^2+7}{2n+3} = \frac{1}{4}\left(2n - 3 + \frac{37}{2n+3}\right)$. So if m is an integer, so is $\frac{37}{2n+3}$. Hence $2n+3 \in \{1, -1, 37, -37\}$.

$$\begin{aligned}
2n+3 = 1 &\Rightarrow n = -1,\ m = 8 \\
2n+3 = -1 &\Rightarrow n = -2,\ m = -11 \\
2n+3 = 37 &\Rightarrow n = 17,\ m = -8 \\
2n+3 = -37 &\Rightarrow n = -20,\ m = -11
\end{aligned}$$

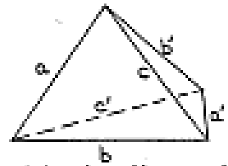
7.2 There are $7! = 5040$ numbers on the list. There are $6! = 720$ which begin with 1 and $6! = 720$ which begin with 2. So the 1994^{th} number begins with 3. Since $1994 = 2\cdot 720 + 554$, we want the 554^{th} number beginning with 3.

Now $6! = 120$ numbers begin with 31, 120 begin with 32, 120 begin with 34 and 120 begin with 35. So the 554^{th} number begins with 36. Since $554 = 4\cdot 120 + 74$, we want the 74^{th} number beginning with 36. Now $4! = 24$ numbers begin with 361, 24 begin with 362 and 24 begin with 364. So the 74^{th} number begins with 365. Since $74 = 3\cdot 24 + 2$, we want the second number on the list beginning with 365, which is 3651274.

7.3

$$\begin{aligned}
\cos 20^\circ \cos 40^\circ \cos 80^\circ &= \cos 20^\circ \cdot \frac{1}{2}(\cos 120^\circ + \cos 40^\circ) \\
&= -\frac{1}{4}\cos 20^\circ + \frac{1}{2}\cos 20^\circ \cos 40^\circ \\
&= -\frac{1}{4}\cos 20^\circ + \frac{1}{4}(\cos 60^\circ + \cos 20^\circ) \\
&= \frac{1}{8}
\end{aligned}$$

7.4 If the sides of the tetrahedron are a, b, c, a', b', c' as shown, then



$$\begin{aligned} a+b+c &= a+b'+c' \\ a'+b+c' &= a'+b'+c \end{aligned}$$

and so $b-b'=c'-c$ and $b-b'=c-c'$, giving $b=b'$, $c=c'$, $a=a'$. So the triangles all have sides of length a, b, c, and are congruent.

7.5 If n is even, then n^4+4^n is composite. If $n=1$, then $n^4+4^n=5$ which is prime. If $n>1$ is odd, then

$$\begin{aligned} n^4+4^n &= (n^2+2^n)^2-2\cdot 2^n\cdot n^2 \\ &= (n^2+2^n)^2-\left(2^{\frac{n+1}{2}}n\right)^2 \\ &= \left(n^2+2^n-n\cdot 2^{\frac{n+1}{2}}\right)\left(n^2+2^n+n\cdot 2^{\frac{n+1}{2}}\right) \end{aligned}$$

This number is composite, since by completing the square we get $n^2+2^n-n\cdot 2^{\frac{n+1}{2}}=\left(n-2^{\frac{n-1}{2}}\right)^2+2^n-2^{n-1}>1$.

8.2 Solutions to the Stellenbosch Tests

Solutions: Test I

1.

$$\begin{aligned} & \frac{b}{1+b}+\frac{c}{1+c}-\frac{a}{1+a} \\ = & \frac{b(1+a)(1+c)+c(1+a)(1+b)-a(1+b)(1+c)}{(1+a)(1+b)(1+c)} \\ = & \frac{abc+2bc+b+c-a}{(1+a)(1+b)(1+c)} \\ > & \frac{abc+2bc}{(1+a)(1+b)(1+c)} \\ > & 0 \end{aligned}$$

2. This is an example of a telescoping series:

$$\begin{aligned} & \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)} \\ = & \frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{n}-\frac{1}{n+1} \\ = & 1-\frac{1}{n+1} \end{aligned}$$

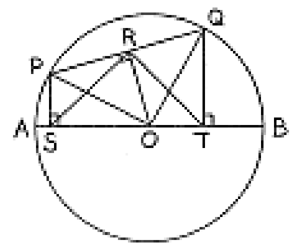
3. $\phi(100)=40$, and $(3,100)$, so by Euler's theorem [4] $3^{40}\equiv_{100}1$. In fact, we want to find the smallest n for which $3^n\equiv_{100}1$. Certainly $n|40$. Since $3^4\equiv_{10}1$, $4|n$. Hence $n\in\{8,\ 20,\ 40\}$. Since $3^8=81^2\equiv_{100}61$, $n\neq 8$. But $3^{20}\equiv_{100}61\cdot 61\cdot 81\equiv_{100}1$. Hence $3^{1996}\equiv_{100}3^{16}\equiv_{100}61\cdot 81\cdot 27\equiv_{100}7$. Thus the answer is 07.

4. If $m=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$, with p_1, p_2, $\ldots$, p_n distinct primes, then the number of divisors of m is $(k_1+1)(k_2+1)\cdots(k_n+1)$. Since $30=2\cdot 3\cdot 5$, $n\leq 3$.

[4] Euler's theorem says that if $(a,n)=1$ then $a^{\phi(n)}\equiv_n 1$. Here $\phi(n)$ is the amount of numbers both less than and relatively prime to n. It can be shown that $\phi(n)=n\cdot\prod\left(1-\frac{1}{p}\right)$; here the product is taken over all primes less than or equal to n. For further details see the South African Mathematical Society Olympiad Training Notes no.2, 'Topics in Number Theory', by Valentin Goranko.

- For $n = 1$ the smallest value of m is 2^{29};
- for $n = 2$ we try $2^{14} \cdot 3$, $2^9 \cdot 3^2$, $2^5 \cdot 3^4$, $2^4 \cdot 3^5$, $2 \cdot 3^{14}$;
- for $n = 3$ the smallest value of m is $2^4 \cdot 3^2 \cdot 5 = 720$.

5. Suppose O is the centre of the circle. Then $OR \perp PQ$ and so we have that $PROS$ and $ROTQ$ are both cyclic. Therefore $\angle RSO = \angle RPO = \angle RQO = \angle RTO$, and ΔRST is isosceles.



If $2PQ = AB$, then $PQ = OP = OQ$ and hence $\angle RST = \angle RTS = 60°$. Conversely, if ΔRST is equilateral, $\angle RST = 60°$ and hence $\angle RPO = 60°$. So ΔPOQ is equilateral and $PQ = PO$. So $2PQ = 2PO = AB$.

6. We use Jensen's inequality [5] with the concave function $f(x) = \sqrt[3]{x}$:

$$\begin{aligned} 2 + \sqrt[3]{7} &= \sqrt[3]{8} + \sqrt[3]{7} \\ &\leq 2\sqrt[3]{\frac{8+7}{2}} \\ &= \sqrt[3]{8 \cdot \frac{8+7}{2}} \\ &= \sqrt[3]{60} \end{aligned}$$

7. First note that if $2^m - 1 = 3^n$ then m must be even, and if $2^m + 1 = 3^n$ then m must be odd. (Reduce each equation $\equiv_3$.)

Case 1: $2^m - 1 = 3^n$.
Using difference of squares we get $\left(2^{m/2} - 1\right)\left(2^{m/2} + 1\right) = 3^n$.

[5] See the South African Mathematical Society Olympiad Training Notes no.3, 'Inequalities for the Olympiad Enthusiast', by Graeme West.

(This is a good idea because m is even.) Thus each bracket is a power of three. If we assume that each bracket is > 1, however, we get from our preliminary observation that $\frac{m}{2}$ is even (since $\left(2^{m/2} - 1\right)$ is a power of three) and $\frac{m}{2}$ is odd (since $\left(2^{m/2} + 1\right)$ is a power of three, a contradiction. Thus $\left(2^{m/2} - 1\right) = 1$, and we get $m = 2$, $n = 1$.

Case 2: $2^m + 1 = 3^n$.
If $m = 1$ then $n = 1$. So we can suppose $m > 1$. Then by reducing the equation $\equiv_4$ we get that n is even. Thus $2^m = 3^n - 1 = \left(3^{n/2} - 1\right)\left(3^{n/2} + 1\right)$. If we put $3^{n/2} - 1 = 2^\alpha$ and $3^{n/2} + 1 = 2^\beta$ then by taking differences we get $2 = 2^\beta - 2^\alpha$, and deduce that $\alpha = 1$, and thus $n = 2$, $m = 3$.

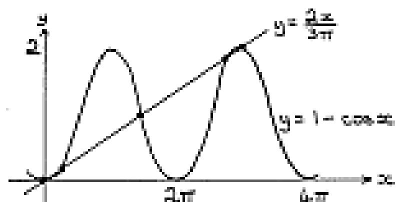
Solutions: Test II

1. $f(x) = x$ clearly satisfies the conditions. Suppose that $f(x) < x$ for some $x \in \mathbb{R}$. Then $x = f(f(x)) \leq f(x)$, a contradiction. Similarly $x < f(x)$ leads to a contradiction. So $f(x) = x$ is the only solution.

2. We get that $n \equiv_3 0$ and $m \equiv_5 1$, so put $n = 3\alpha$ and $m = 5\beta + 1$. Then $3(5\beta + 1) + 5(3\alpha) = 1008$, or $\alpha + \beta = 67$. Hence $n = 3\alpha$ and $m = 5(67 - \alpha) + 1 = 336 - 5\alpha$ for $1 \leq \alpha \leq 67$.

3. Suppose $p(0) = q$, a prime. Then $p(x)$ is a multiple of q whenever x is a multiple of q, and so must be q itself. Thus $p(x) = q$ for x any multiple of q. From the Fundamental Theorem of Algebra we know that a polynomial cannot take on a specific value infinitely many times unless it is constant. Hence $p(x) \equiv q$. So the required set is all the constant polynomials with prime values.

4. Draw the graphs of $y = 1 - \cos x$ and $y = \frac{2}{3\pi}x$.



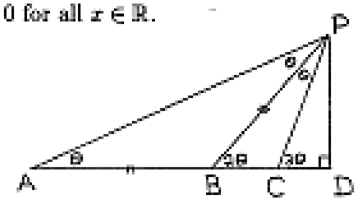
These graphs intersect at $(0, 0)$ and at $(3\pi, 2)$, and there are no points of intersection for $x < 0$ or $x > 3\pi$. Since the tangent to $y = 1 - \cos x$ is horizontal at $(0, 0)$ and at $(3\pi, 2)$, there are two further points of intersection, each 'close' to the previous two mentioned points. Finally, there is clearly a fifth intersection point more or less in the middle, and no others.

5. Suppose $x = 0.a_{11}a_{22}a_{33}\ldots$ is rational. Let $b_i = 9 - a_{ii}$, and let $y = 0.b_1b_2b_3\ldots$. Then $x + y = 1$, so y is also rational. Hence $y = 0.a_{m1}a_{m2}a_{m3}\ldots$ for some $m \in \mathbb{N}$. Then $b_m = a_{mm} = 9 - a_{mm}$, or $2a_{mm} = 9$, a contradiction.

Solutions: Test III

1. Putting $y = 0$, we get $f(x) = 0$ for all $x \in \mathbb{R}$.

2.



$$\begin{aligned}\frac{AB}{BC} &= \frac{BP}{BC} \\ &= \frac{\sin 3\theta}{\sin \theta} \\ &= \frac{3\sin\theta - 4\sin^3\theta}{\sin\theta} \\ &= 3 - 4\sin^2\theta\end{aligned}$$

Now $3 - 4\sin^2\theta \to 3$ as $\theta \to 0$. Finally, $\theta < \frac{\pi}{6} \Rightarrow \sin\theta < \frac{1}{2}$, and so $3 - 4\sin^2\theta > 2$.

3. Put $a = x - y$, $b = y - z$, $c = z - x$. Then $a + b + c = 0$. Hence

$$\begin{aligned}30 &= a^3 + b^3 + c^3 \\ &= a^3 + b^3 + (-a - b)^3 \\ &= -3a^2b - 3ab^2 \\ &= -3ab(a + b) \\ &= 3abc\end{aligned}$$

Thus the given system has been reduced to

$$\begin{aligned}0 &= a + b + c \\ 10 &= abc\end{aligned}$$

and so a, b, $c \in \{\pm 1, \pm 2, \pm 5, \pm 10\}$. By symmetry we may suppose that $|a| \geq |b| \geq |c|$, and it is easily seen there are no solutions.

4. We have in cyclic notation that $\phi^2 = (1, 7, 4, 2)(3, 6, 5, 8)$. Thus in ϕ we will have

$$1 \to x \to 7 \to y \to 4 \to z \to 2 \to w \to 1$$

and

$$3 \to a \to 6 \to b \to 5 \to c \to 8 \to d \to 3.$$

Putting $x = 7$ leads to the contradiction $7 \to 7$ and $7 \to 4$. Similarly putting $x = 4$ or $x = 2$ leads to contradictions. Thus $x \in \{3, 6, 5, 8\}$, and each possibility leads by combining the above 'flows' to exactly one solution. They are: $(1,3,7,6,4,5,2,8,1)$, $(1,6,7,5,4,8,2,3,1)$, $(1,5,7,8,4,3,2,6,1)$, $(1,8,7,3,4,6,2,5,1)$.

Solutions: Test IV

1. The prince must move $n-1$ moves up and $m-1$ moves right, for a total of $m+n-2$ moves. Of the $m+n-2$ moves, the prince simply needs to choose which $n-1$ are up. Hence the number of possible ways is $\binom{m+n-2}{n-1}$.

2. It is easy to verify inductively that $f(x+\alpha 3) = (x+(\alpha-1)3)^2 - (x+(\alpha-2)3)^2 + \cdots + (-1)^{\alpha+1}x^2 + (-1)^{\alpha} f(x)$ for $\alpha \geq 2$. Hence, with $\alpha = 25$, we get

$$\begin{aligned} & f(94) \\ = \; & 91^2 - 88^2 + 85^2 - 82^2 + \cdots - 22^2 + 19^2 - f(19) \\ = \; & 3(91+88) + 3(85+82) + \cdots + 3(25+22) + 19^2 - 94 \\ = \; & 3 \cdot \frac{1}{2} \cdot 24 \cdot (91+22) + 19^2 - 94 \\ = \; & 4335 \end{aligned}$$

3. Suppose $H(n) = N$ where $N \in \mathbb{N}$. Now choose $k \in \mathbb{N}$ such that $2^k \leq n < 2^{k+1}$. Consider the l.c.m. of $\{1, 2, \ldots, n\} \setminus \{2^k\}$, and write it as $2^{k-1}P$ where P is odd. We then have $2^{k-1}P \cdot H(n) = N2^{k-1}P$. The left hand side consists of the sum of various integers, and also $2^{k-1}P \cdot \frac{1}{2^k} = \frac{P}{2}$, while the right hand side is an integer. This is a contradiction.

4. We aim to use the double angle formula

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

The only problem could be if $\tan\alpha\tan\beta = 1$, but if this happened, then $\tan\alpha + \tan\beta + \tan\gamma = \tan\gamma$, and so $\tan\alpha + \tan\beta = 0$. But $\tan\alpha\tan\beta = 1$, $\tan\alpha + \tan\beta = 0$ is impossible (they have the same sign from the first equation but different signs from the second).

Returning to the problem, we thus have that $\tan\alpha + \tan\beta = \tan\gamma(\tan\alpha\tan\beta - 1)$, and so $\tan(\alpha+\beta) = -\tan\gamma = \tan(-\gamma)$. Hence $\alpha + \beta = -\gamma + n\pi$, since the tangent function has period π. Thus $\alpha + \beta + \gamma = n\pi$.

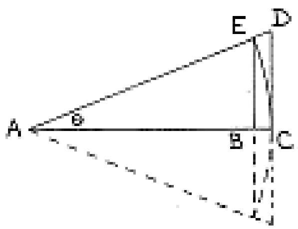
Solutions: Test V

1. There are $\left\lfloor \frac{n}{p} \right\rfloor$ members from $\{1, 2, \ldots, n\}$ which are divisible by p. We seek all the products of k numbers from $\{1, 2, \ldots, n\}$ which contain at least one representative from the abovementioned set. This equals the number of arbitrary products less the number of products which have no representatives from the set at all, that is, $\binom{n}{k} - \binom{n-\lfloor \frac{n}{p} \rfloor}{k}$.

2. (a) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\frac{t}{\sqrt{1+t^2}}\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$.

   $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \frac{2t}{1-t^2}$.

   (b) The diagram indicates $1/1995^{th}$ of the situation. We then consider just half of that, and mark the diagram as illustrated. Furthermore by scaling we may suppose that $AC = 1$. Hence the arc EC has length θ, and it suffices to prove that $\sin\theta + \tan\theta \geq 2\theta$.



$$\begin{aligned}
\sin\theta + \tan\theta &= \frac{2t}{1+t^2} + \frac{2t}{1-t^2} \\
&= \frac{2t(1-t^2) + 2t(1+t^2)}{(1+t^2)(1-t^2)} \\
&= \frac{4t}{1-t^4} \\
&> 4t \\
&= 4\tan\frac{\theta}{2} \\
&> 4\frac{\theta}{2} \\
&= 2\theta
\end{aligned}$$

3. This is a slight variation on IMO 1983 Question 1, which appears in as Monthly Problem #9 in this book.

4. Note that $a_1 = 1$ and $a_2 = 3$. We analyse the value of a_n depending on whether n is even or odd.

   n is odd:
   Then $a_{n+2} = 2a_{n+1} = 4a_n + 2$, and so by calculation we get that $a_1 = 1$, $a_3 = 6 = 1+5$, $a_5 = 26 = 1+5+5\cdot 4$, $a_7 = 106 = 1+5+5\cdot 4+5\cdot 4^2$. This suggests that $a_{2n+1} = 1 + \frac{5}{3}\left[2^{2n}-1\right]$ for $n \geq 1$, and this is easily verified using induction.

   Hence by change of variable we get $a_m = 1 + \frac{5}{3}\left[2^{m-1}-1\right] = \frac{5}{6}2^m - \frac{2}{3}$ for odd $m \geq 3$; in fact this works for $m = 1$ as well.

   n is even:
   Then $a_{n+2} = 2a_{n+1} + 1 = 4a_n + 1$, and so by calculation we get that $a_2 = 3$, $a_4 = 13 = 3+10$, $a_6 = 53 = 3+10+10\cdot 4$, $a_8 = 213 = 3+10+10\cdot 4+10\cdot 4^2$. This suggests that $a_{2n+2} = 3 + \frac{10}{3}\left[2^{2n}-1\right]$ for $n \geq 0$, and this is again easily verified using induction.

   Hence by change of variable we get $a_m = 3 + \frac{10}{3}\left[2^{m-2}-1\right] = \frac{5}{6}2^m - \frac{1}{3}$ for even $m \geq 2$.

   Hence $a_m = \frac{5}{6}2^m + \frac{1}{6}(-1)^m - \frac{1}{2}$ for all $m \in \mathbb{N}$.

8.3 Solutions to the monthly problem sets

1. A possible choice is the set $\{2,\ 2^23,\ 2^23^25,\ 2^23^25^27,\ \ldots\}$.

2. We may suppose, without loss of generality, that $|x| \geq |y| \geq |z|$. The problem is of course trivial if $|x| = 0$, so suppose that that is not the case. Now divide throughout by $|x|$. What we get is the equivalent problem: show that if $u,\ v \in \mathbb{R}$ and $|u|,\ |v| \leq 1$ then

$$1 + |u| + |v| - |1+u| - |u+v| - |v+1| + |1+u+v| \geq 0.$$

(Put $u = y/x$ and $v = z/x$.) But then, $|1+u| = 1+u$ and $|1+v| = 1+v$, and so, rearranging, we are required to show that

$$|u| + |v| - |u+v| + |1+u+v| - (1+u+v) \geq 0.$$

That's obvious!

3. Let O_1 and ρ be the centre and the radius of k_0. Clearly

$$CI = \frac{r}{\sin\frac{C}{2}},\quad CO_1 = \frac{\rho}{\sin\frac{C}{2}},\quad IO_1 = \frac{\rho - r}{\sin\frac{C}{2}}$$

and so $\frac{IO_1}{CO_1} = \frac{\rho - r}{\rho} = 1 - \frac{r}{\rho}$. By using the fact that $\angle OCI = \frac{|A-B|}{2}$ and by making use of the cosine law at $\angle COO_1$ we get that

$$\rho \cdot \cos^2\left(\frac{C}{2}\right) = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = r.$$

Thus we have that

$$\frac{IO_1}{CO_1} = \sin^2\left(\frac{C}{2}\right) = \frac{\rho^2}{CO_1^2}$$

and so $IO_1 \cdot CO_1 = \rho^2$. But this shows that CO_1 is perpendicular to DE and meets it at I, and therefore I is the midpoint of DE.

(Proposed at the 1993 IMO)

4. We have that

$$\begin{aligned} r &= 4R\ \sin\frac{A}{2}\ \sin\frac{B}{2}\ \sin\frac{C}{2} \end{aligned}$$

$$\begin{aligned} &= 4R\ \sin\frac{A}{2}\ \frac{\cos\frac{B-C}{2} - \cos\frac{B+C}{2}}{2} \\ &= 4R\ \sin\frac{A}{2}\ \frac{\cos\frac{B-C}{2} - \sin\frac{A}{2}}{2} \\ &= 2R\ \sin\frac{A}{2}\ \cos\frac{B-C}{2} - 2R\ \sin^2\frac{A}{2} \end{aligned}$$

Now think of $\sin\frac{A}{2}$ as a variable, call it x. Then

$$2Rx^2 - 2R\cos\frac{B-C}{2}x + r = 0$$

and so the discriminant of this quadratic in x is non-negative, that is

$$4R^2\cos^2\frac{B-C}{2} - 8Rr \geq 0$$

and so

$$\cos^2\frac{B-C}{2} \geq \frac{2r}{R}.$$

5. Since $0 \leq x_i,\ 0 \leq y_i,\ x_i + y_i = 1$ for $i = 1,\ 2,\ \ldots,\ n$ we can imagine that we have n coins which are (un)balanced in such a way that the probability of a head with the i^{th} coin is x_i and the probability of a tail is y_i.

Then the first quantity in the left hand side is the probability that there is at least one tail in each of m sets of tosses of all n coins. The second quantity in the left hand side is the probability that each coin has been a head at least once in m sets of tosses of all n coins. Now either of these events must have occurred (for if the first has never occurred, then on one of the sets of tosses all the coins were heads, and so the second event occurred) and so the sum of their probabilities is ≥ 1.

6. There is a winning strategy for the first player. Since the game has a finite number of moves, there must be a winner. Therefore there must be a winning strategy for either the first player or the second player. We show by contradiction that this strategy cannot be possessed by the second player.

Suppose it is. Notice that if the number 1 is not 'played' by the first player on the first move of any game then it can never be played. So let the first player, on their first move, play the number 1. This is equivalent to passing the first move, and now the second player finds themselves in the first position. No matter which move the second player now makes, the first player applies the winning strategy (of the second player) as if he were the second player. Thus the first player wins, which contradicts the fact that the second player has a winning strategy.

7. We call an ordered pair (n, m) a solution pair if $1 \leq m,\ n \leq 1981$ and $(n^2 - nm - m^2)^2 = 1$.

If $m = 1$ then $(1, 1)$ and $(2, 1)$ are the only solution pairs. For any solution pair (n_1, n_2) with $n_1 > 1$ we have $n_1(n_1 - n_2) = n_2^2 \pm 1 > 0$ so that $n_1 > n_2$. Define $n_3 = n_1 - n_2$; then $n_1 = n_2 + n_3$ and

$$\begin{aligned} 1 &= (n_1^2 - n_1 n_2 - n_2^2)^2 \\ &= ((n_2 + n_3)^2 - (n_2 + n_3)n_2 - n_2^2)^2 \\ &= (-n_2^2 + n_2 n_3 + n_3^2)^2 \\ &= (n_2^2 - n_2 n_3 - n_3^2)^2 \end{aligned}$$

so (n_2, n_3) is also an solution pair. In the same manner as before we have $n_2 > n_3$.

Inductively we have a necessarily finite sequence $n_1 > n_2 > n_3 > \cdots$ such that $n_{i+1} = n_{i-1} - n_i$ and where (n_i, n_{i+1}) is a solution pair for all i. The sequence terminates when $n_i = 1$. Since $(n_{i-1}, 1)$ is a solution pair, and $n_{i-1} > 1$, $n_{i-1} = 2$ must hold. Running it backwards from $(2, 1)$ we determine uniquely the Fibonacci sequence 1, 2, 3, 5, 8, 13, ..., 987, 1597.

Hence the maximum possible value of $m^2 + n^2$ is $1597^2 + 987^2$

(IMO 1981 Question 3)

8. The given expression is rewritten as

$$\begin{aligned} x^3 - 3xy^2 + y^3 &= (y - x)^3 - 3x^2 y + 2x^3 \\ &= (y - x)^3 - 3(y - x)x^2 + (-x)^3 \end{aligned}$$

Hence, if a pair (x, y) is a solution, then so is the pair $(y - x, -x)$. By a similar algebraic manipulation, we can show that $(-y, x - y)$

is then also a solution. Moreover, these three pairs are distinct, for if two were identical, then we would easily conclude that $x = y = 0$ and this is excluded by the condition that $n > 0$.

To show that there are no integer solutions to $x^3 - 3xy^2 + y^3 = 2981$ we first consider the solutions $\equiv_3$; the equation reduces to $x^3 + y^3 \equiv_3 -1$. This in turn reduces to $x + y \equiv_3 -1$. Thus we have three cases :-
(i) $x \equiv_3 0,\ y \equiv_3 -1$, (ii) $x \equiv_3 1,\ y \equiv_3 1$, (iii) $x \equiv_3 -1,\ y \equiv_3 0$.
Then in case (i), putting $x = 3m$ and $y = 3n - 1$, and substituting in the original equation, we get

$$-1 \equiv_9 (3m)^2 - 3(3m)(3n - 1)^2 + (3n - 1)^3 \equiv_9 2891 \equiv_9 2$$

which is a contradiction. Case (ii) cannot occur, since it was shown that if (x, y) is a solution then so is $(y - x, -x)$; so case (ii) leads into case (i). Similarly case (iii) leads into case (ii).

(IMO 1982 Question 4)

9. We have $f^2(y) = y\,f(1)$, and since $f(1) \neq 0$, it follows that f is bijective. Hence there is a value y such that $f(y) = 1$. This together with $x = 1$ in (i) gives

$$f(1 \cdot 1) = f(1) = y f(1)$$

and since $f(1) > 0$ by hypothesis, it follows that $y = 1$, and so $f(1) = 1$. When we set $y = x$ in (i) we get

$$f(xf(x)) = xf(x)$$

for all $x > 0$. Hence $xf(x)$ is a fixed point of f.

Now if x and y are fixed points of f then (i) implies that

$$f(xy) = yx$$

so xy is also a fixed point of f. Thus the set of fixed points is closed under multiplication. Furthermore, if x is fixed point then

$$1 = f(1) = f\left(\frac{1}{x} f(x)\right) = x\,f\left(\frac{1}{x}\right)$$

and so $f\left(\frac{1}{x}\right) = \frac{1}{x}$, that is, $\frac{1}{x}$ is a fixed point. The set of fixed points is closed under inversion.

Thus if there are any fixed points besides 1, then either it or its inverse is bigger than 1 (and is a fixed point), and then the powers of this number become arbitrarily big and are all fixed points. This contradicts (ii).

Thus 1 is the only fixed point, and since $xf(x)$ is a fixed point for every x, we get that $1 = xf(x)$, or $f(x) = \frac{1}{x}$.

This checks.

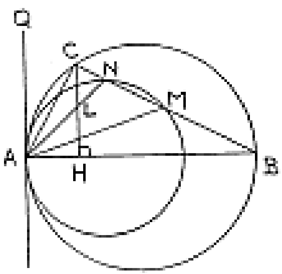
(IMO 1983 Question 1)

10. We colour the edges of the triangle according to the following rule: an edge is coloured blue if it is the shortest edge in some triangle, and then the remaining edges are coloured red. By a well known graph theory result, there exists a monochromatic triangle. By construction this cannot be a red triangle. The longest side in a monochromatic blue triangle satisfies the requirement.

11.

$$\begin{aligned}
& \sin 2\alpha + \sin 2\beta + \sin 2\gamma \\
= {} & \sin 2\alpha + \sin 2\beta - \sin(2\alpha + 2\beta) \\
= {} & 2\sin(\alpha+\beta)\cos(\alpha-\beta) - 2\sin(\alpha+\beta)\cos(\alpha+\beta) \\
= {} & 2\sin(\alpha+\beta)\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right] \\
= {} & 2\sin(\alpha+\beta)\cdot 2\sin\alpha\sin\beta \\
= {} & 4\sin\alpha\sin\beta\sin\gamma
\end{aligned}$$

12. Since A lies on both circles, A is the point of tangency. Let QA be the common tangent line and let L be the point of intersection of AN and CH.
Then $\angle ACH = \angle QAC = \angle CBA$ by the tangent-chord theorem, and so $\triangle ACH \,|||\, \triangle CBH \,|||\, \triangle ABC$.
Also $\angle ALH = \angle QAN = \angle NMA$, and thus $\triangle ALH \,|||\, \triangle AMC$.



What now? Ratios involving CB and MC are known, ratios involving CH and LH are sought. So we get

$$\frac{HC}{BC} = \frac{AH}{AC} = \frac{LH}{MC},$$

that is,

$$\frac{HC}{LH} = \frac{BC}{MC} = 2.$$

(Bulgarian National Olympiad 1993)

13. Let $I = \{1, 2, \ldots, 1000\}$; arithmetic with I is done modulo 1000. (So for example, the number after 1000 is 1.)

We need only consider the case $a > 1$, because in the case $a < -1$ we can replace a with $-a$ and x_i with $-x_i$ for every $i \in I$.

Now $x_{i-1}^2 = ax_i + 1 \Rightarrow 0 \le ax_i + 1 \Rightarrow -1 \le ax_i \Rightarrow \frac{-1}{a} \le x_i \Rightarrow -1 < x_i$ for all $i \in I$.

There are two possible cases:

(a) There exists $i \in I$ such that $x_i \ge 0$. Then $x_i \ge 0 \Rightarrow ax_i \ge 0 \Rightarrow ax_i + 1 \ge 1 \Rightarrow x_{i-1}^2 \ge 1 \Rightarrow x_{i-1} \ge 1$ (because $x_{i-1} > -1$). Now, by iterating, we see that $x_i \ge 1$ for all $i \in I$.

(b) $-1 < x_i < 0$ for all $i \in I$.

Let's analyse these cases:

(a) $x_i \ge x_j \Rightarrow x_i^2 \ge x_j^2 \Rightarrow ax_{i+1} + 1 \ge ax_{j+1} + 1 \Rightarrow x_{i+1} \ge x_{j+1}$. So, let x_1 be the largest of all the x_i. Then from this calculation, $x_1 \ge x_2 \Rightarrow x_2 \ge x_3$, then $x_2 \ge x_3 \Rightarrow x_3 \ge x_4$, etc. So $x_1 \ge x_2 \ge \cdots \ge x_{1000} \ge x_1$, and so they are all equal to x where $x > 1$ is the solution to $x^2 = ax + 1$, that is,

$$x_1 = x_2 = \cdots = x_{1000} = \frac{a + \sqrt{a^2+4}}{2}$$

(b) $x_i \ge x_j \Rightarrow x_i^2 \le x_j^2 \Rightarrow ax_{i+1} + 1 \le ax_{j+1} + 1 \Rightarrow x_{i+1} \le x_{j+1}$. So, let x_1 be the largest of all the x_i. Then from this calculation, $x_1 \ge x_3 \Rightarrow x_2 \le x_4 \Rightarrow x_3 \ge x_5 \Rightarrow x_4 \le x_6, \ldots$. Thus

$$x_1 \ge x_3 \ge x_5 \ge \cdots \ge x_{999} \ge x_1$$

$$x_2 \le x_4 \le x_6 \le \cdots \le x_{1000} \le x_2$$

and the problem reduces to solving

$$\begin{aligned} x_O^2 &= a x_E + 1 \\ x_E^2 &= a x_O + 1 \end{aligned}$$

Then $x_O^2 - x_E^2 = a(x_E - x_O)$ and so

$$0 = (x_O - x_E)(x_O + x_E + a)$$

- $x_O = x_E$: Then $x_1 = x_2 = \cdots = x_{1000} = \frac{a-\sqrt{a^2+4}}{2}$.
- $x_E = -a - x_O$: then $x_O^2 = a(-a - x_O) + 1$ and so $0 = x_O^2 + a x_O + a^2 - 1$. Then

$$x_1 = x_3 = \cdots = x_{999} = \frac{-a \pm \sqrt{4 - 3a^2}}{2}$$

$$x_2 = x_4 = \cdots = x_{1000} = \frac{-a \mp \sqrt{4-3a^2}}{2}.$$

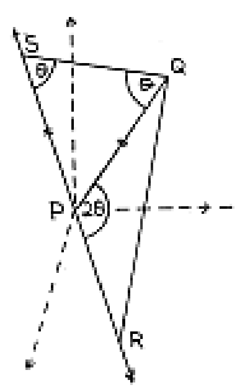
Here we must assume that $4 - 3a^2 \ge 0$, that is, $\frac{2}{\sqrt{3}} \ge a$.

(Proposed at the 1993 IMO)

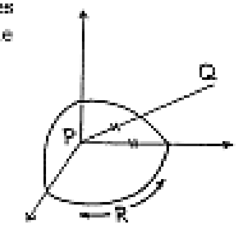
14. Suppose that the line through P which contains R has been specified, and let $\angle RPQ = 2\theta$. (Since the line is specified, θ is a fixed quantity until further notice.) Extend RP to S so that $PS = PQ$. The ratio of interest is then

$$\frac{QP + PR}{QR} = \frac{SR}{QR} = \frac{\sin \angle SQR}{\sin \theta}$$

which is maximized if $\angle SQR = 90°$. Then $\triangle QPR$ is isosceles. Thus, on the specified line, R is positioned so that $QP = PR$.



What we have shown is that for maximization we must have $QP = PR$. Let θ no longer be fixed. As θ varies so R varies in the plane π along the circle of radius $|QP|$ and centre P.



We must decide for what value of θ the ratio will be maximized. But this ratio has been seen to be $\frac{\sin \angle SQR}{\sin \theta} = \frac{1}{\sin \theta}$. This is maximized when $\sin \theta$ is minimized. But θ is at most $90°$ and so $\sin \theta$ is minimized when θ is. This occurs when R lies directly below Q (or more precisely, R is the orthogonal projection of Q onto π).

There is an exceptional special case, namely, when Q lies directly above P. In this case θ is constantly $45°$, and R is any point on the circle centred at P of radius $|QP|$.

(IMO 1979 Question 4)

15.

$$\begin{aligned} & \cos 2\alpha + \cos 2\beta + \cos 2\gamma \\ =\ & \cos 2\alpha + \cos 2\beta + \cos(2\alpha + 2\beta) \\ =\ & 2\cos(\alpha+\beta)\cos(\alpha-\beta) + 2\cos^2(\alpha+\beta) - 1 \\ =\ & 2\cos(\alpha+\beta)\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right] - 1 \\ =\ & 2\cos(\alpha+\beta)\cdot 2\cos\alpha\cos\beta - 1 \\ =\ & 2\cdot -\cos\gamma \cdot 2\cos\alpha\cos\beta - 1 \\ =\ & -4\cos\alpha\cos\beta\cos\gamma - 1 \end{aligned}$$

16. We are required to show that

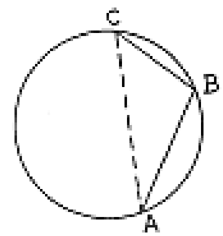
$$\frac{\ln(f(x)) + \ln(f(y))}{2} \le \ln\left[f\left(\frac{x+y}{2}\right)\right]$$

and by standard properties of the logarithm functions, this is the same as $\ln(f(x)\cdot f(y)) \le \ln\left[f\left(\frac{x+y}{2}\right)\right]^2$. But by the monotonicity of ln, this would follow if we can establish that $f(x)\cdot f(y) \le \left[f\left(\frac{x+y}{2}\right)\right]^2$. But this is easy:

$$\sqrt{f(x)\cdot f(y)} \le \frac{f(x)+f(y)}{2} \le f\left(\frac{x+y}{2}\right).$$

The first inequality is the arithmetic-geometric mean inequality, and the second comes from the concavity of f.

17. Suppose one of the angles of the polygon is $\ge 90°$, as illustrated. Then

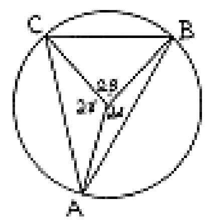


$$\begin{aligned} & AC^2 \\ = \; & AB^2 + BC^2 - 2\,AB\cdot AC\cdot\cos B \\ \ge \; & AB^2 + BC^2 \end{aligned}$$

since $\cos B \le 0$. Hence, deleting AB and BC and drawing in AC, we get a greater sum of squares.

Thus all angles are acute. It now follows that the polygon is a triangle and that the centre of the circle is inside the triangle.

Note that



$$\begin{aligned} AB^2 &= 1+1-2\cos 2\alpha \\ &= 2-2\cos 2\alpha, \end{aligned}$$

and similarly for BC^2 and CA^2. Thus, our problem has been reduced to the following: maximize

$$6 - 2\cos 2\alpha - 2\cos 2\beta - 2\cos 2\gamma$$

subject to $0 < \alpha,\ \beta,\ \gamma < \pi$ and $\alpha+\beta+\gamma = \pi$. But from #15 we have that this quantity is $8 + 8\cos\alpha\cos\beta\cos\gamma$ i.e. we just need to maximize $\cos\alpha\cos\beta\cos\gamma$. Clearly, then, $0 < \alpha,\ \beta,\ \gamma < \frac{\pi}{2}$.

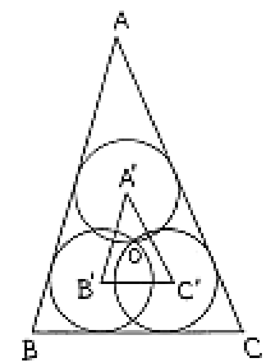
Now $\cos x$ is concave on $\left(0, \frac{\pi}{2}\right)$ and then, from the previous exercise, so is $\ln(\cos x)$. Since the ln function is increasing, maximization of $\cos\alpha\cos\beta\cos\gamma$ occurs at the same time as that of $\ln(\cos\alpha\cos\beta\cos\gamma)$. So:-

$$\begin{aligned} \ln(\cos\alpha\cos\beta\cos\gamma) &= \ln\cos\alpha + \ln\cos\beta + \ln\cos\gamma \\ &\le 3\ln\cos\frac{\alpha+\beta+\gamma}{3} \\ &= 3\ln\cos\frac{\pi}{3}. \end{aligned}$$

Here we used Jensen's inequality. Maximization occurs uniquely when $\alpha = \beta = \gamma = \frac{\pi}{3}$, that is, when we have an equilateral triangle, and the maximized amount is $8 + 8\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = 9$.

(Proposed at an IMO)

18. To solve this problem we use the properties of homothetisms. The basic property of homothetic figures is that corresponding points lie on a line passing through the point of homothety. We will indicate homothetic figures with a $\sim$.



Refer to the figure. It is clear that $AC \parallel A'C'$, etc., and so $\Delta A'B'C' \sim \Delta ABC$, and the point of homothety is the common incentre of the two triangles.

We are asked to show that the incentre of ΔABC, the point O and the circumcentre of ΔABC all lie on a straight line. But the incentre of ΔABC is the incentre of $\Delta A'B'C'$. By the property of homothetisms, the (common) incentre and the two circumcentres lie on a straight line (since the incentre is the point of homothety). But O is the circumcentre of $\Delta A'B'C'$.

How about that!

(IMO 1981 Question 5)

19. Let the fourth root be α. Then it follows from standard facts about the roots of polynomials that

$$p = \tan A + \tan B + \tan C + \alpha$$

$$\begin{aligned} q &= \tan A \tan B + \tan B \tan C + \tan C \tan A \\ &\quad + \alpha(\tan A + \tan B + \tan C) \\ r &= \alpha(\tan A \tan B + \tan B \tan C + \tan C \tan A) \\ &\quad + \tan A \tan B \tan C \\ s &= \alpha \tan A \tan B \tan C \end{aligned}$$

Let $T = \tan A \tan B \tan C = \tan A + \tan B + \tan C$ (this is the key fact about the tangent function) and denote $\Sigma = \tan A \tan B + \tan B \tan C + \tan C \tan A$. Then the equations can be rewritten as

$$\begin{aligned} \Sigma + \alpha &= p \\ T + \alpha\Sigma &= q \\ \alpha T + \Sigma &= r \\ \Sigma\alpha &= s \end{aligned}$$

and so

$$\begin{aligned} T + s &= q \\ \alpha T + p - \alpha &= r \end{aligned}$$

and so $\alpha(q - s) + p - \alpha = r$, and $\alpha = \frac{r-p}{q-s-1}$.

There is a special case when $q - s - 1 = 0$. Then $q = s + 1$ and as a consequence $T = 1$ and $p = r$. Then the original equation is

$$x^4 - px^3 + qx^2 - px + q - 1 = 0$$

which has $\pm i$ as roots. This is impossible, as the original equation had three (and hence, in fact, four) real roots.

(Proposed at an IMO)

20. Suppose $n = 3$. Let the numbers be m, $m-1$, $m-2$. Since $m-1$ and $m-2$ have no common factors, their l.c.m. is $(m-1)(m-2)$, and so $m|(m-1)(m-2) = m^2 - 3m + 2$. Thus $m|2$, a contradiction.

Now suppose $n = 4$. Let the numbers be m, $m-1$, $m-2$, $m-3$. m has no common factors with $m-1$, only (possibly) the factor 2 in common with $m-2$, and only (possibly) the factor 3 in common

with $m-3$. Since all of the factors of m are distributed somehow amongst $m-1$, $m-2$, $m-3$, we get that m is of the form $2^\alpha 3^\beta$.

Suppose $\beta = 0$. Then $2 \nmid m-1$ and $2 \nmid m-3$, showing that $m|m-2$, a contradiction. Similarly $\alpha = 0$ leads to the contradiction $m|m-3$.

Suppose then that α, $\beta \geq 1$. Then, similar to the above, we get that $2^\alpha|m-2$. Put $m - 2 = 2^\alpha x$. Then $2^\alpha x + 2 = 2^\alpha 3^\beta$, or $x + 2^{1-\alpha} = 3^\beta$. Thus $\alpha = 1$. A similar argument involving $m-3$ shows that $\beta = 1$.

Thus there is only one possible solution, namely $\alpha = \beta = 1$, giving $m = 6$, which is indeed a solution.

Now suppose $n > 4$. Let the numbers be m, $m-1$, $\ldots$, $m-(n-3)$, $m-(n-2)$, $m-(n-1)$. (Because $n > 4$, at least this many numbers do actually appear in the list.) We want m to divide the l.c.m. of $m-1$, $\ldots$, $m-(n-3)$, $m-(n-2)$, $m-(n-1)$. We could put $m = (n-1)(n-2)$. Then $n-1|m-(n-1)$ and $n-2|m-(n-2)$, and so $(n-1)(n-2)$ divides the l.c.m. of these two numbers, since they are relatively prime. Then $m = (n-1)(n-2)$ divides the l.c.m. of $m-1$, $\ldots$, $m-(n-3)$, $m-(n-2)$, $m-(n-1)$.

Similarly if we put $m = (n-2)(n-3)$ we get the same result.

Thus the answers are (a) $n \geq 4$ and (b) $n = 4$.

(IMO 1981 Question 4)

21. The proof is by induction. The case $n = 4$ follows from

$$\begin{aligned} & x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 \\ =\ & x_1x_2 + x_2x_3 + x_3(1 - x_1 - x_2 - x_3) + (1 - x_1 - x_2 - x_3)x_1 \\ =\ & x_1 + x_3 - 2x_1x_3 - x_3{}^2 - x_1{}^2 \\ =\ & x_1 + x_3 - (x_1 + x_3)^2 \\ \leq\ & \frac{1}{4} \end{aligned}$$

since the function $x - x^2$ is a quadratic with a maximum of $\frac{1}{4}$.

By the way, a set of numbers $0 \leq x_1, x_2, \ldots, x_n$ with $x_1 + x_2 + \cdots + x_n = 1$ is called a(n n-dimensional) simplex. Suppose the inequality is true for simplexes with n variables ($n \geq 4$), and suppose we are given an $n+1$-dimensional simplex $x_1, x_2, \ldots, x_{n+1}$. Then

$$\begin{aligned} & x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_{n+1} + x_{n+1}x_1 \\ = \; & x_1x_2 + x_2x_3 + \cdots + (x_{n-1} + x_{n+1})x_n + x_{n+1}x_1 \\ \leq \; & x_1x_2 + x_2x_3 + \cdots + (x_{n-1} + x_{n+1})x_n + (x_{n-1} + x_{n+1})x_1. \end{aligned}$$

The expression in the last line is induced by the n-dimensional simplex $y_1, y_2, \ldots, y_n$, where $y_1 = x_1$, $y_2 = x_2$, $y_{n-2} = x_{n-2}$, $y_{n-1} = x_{n-1} + x_{n+1}$, $y_n = x_n$. Thus, by the induction hypothesis, the last line is $\leq \frac{1}{4}$.

22. The proof is via the pigeonhole principle. Let α be any irrational number. Then one of the numbers $\alpha, 2\alpha, \ldots, n\alpha, (n+1)\alpha$ will do the job. Suppose not. Then there is at least one rational number in each row of the following table

$$\begin{array}{cccc} \alpha + a_1 & \alpha + a_2 & \cdots & \alpha + a_n \\ 2\alpha + a_1 & 2\alpha + a_2 & \cdots & 2\alpha + a_n \\ \vdots & \vdots & \vdots & \vdots \\ (n+1)\alpha + a_1 & (n+1)\alpha + a_2 & \cdots & (n+1)\alpha + a_n \end{array}$$

Thus the table contains at least $n+1$ rational numbers. Since it has n columns, at least two of these rationals are in the same column. Thus there exist different indices i and j and an index k such that $i\alpha + a_k$ and $j\alpha + a_k$ are rational. But then the number

$$(i-j)\alpha = (i\alpha + a_k) - (j\alpha + a_k)$$

is also rational, a contradiction.

23. Putting $x = y = 0$, we get $2g(0) = 4g(0)$, and so $g(0) = 0$. Putting $x = 0$, we get $g(y) + g(-y) = 2g(y)$, and so $g(y) = g(-y)$ i.e. g is even.

Let us put $g(1) = a$. Then putting $x = y = 1$, we get $g(2) + g(0) = 2g(1) + 2g(1)$, and so $g(2) = 4a$. Putting $x = 2$, $y = 1$ we get $g(3) + g(1) = 2g(2) + 2g(1)$, which simplifies to g(3) = 9a.

We hypothesise that $g(n) = an^2$ for all $n \in \mathbb{N}$. This is of course proved by induction. Suppose it is true for $1, 2, \ldots, n$. Then putting $x = n$, $y = 1$ we get

$$\begin{aligned} g(n+1) + g(n-1) &= 2g(n) + 2g(1) \\ g(n+1) &= 2an^2 + 2a - a(n-1)^2 \\ &= an^2 + 2an + a \\ &= a(n+1)^2 \end{aligned}$$

In fact, we can improve this result: we can show $g(nx) = n^2g(x)$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$. The induction is identical, and so is omitted. Hence $an^2 = g(n) = g\left(m\,\frac{n}{m}\right) = m^2\, g\left(\frac{n}{m}\right)$ and so $g\left(\frac{n}{m}\right) = a\,\frac{n^2}{m^2}$. Thus $g(x) = ax^2$ for all $x \in \mathbb{Q}^+$. But since the function is even, it follows that $g(x) = ax^2$ for all $x \in \mathbb{Q}$.

It now follows by the continuity of g and the denseness of $\mathbb{Q}$ in $\mathbb{R}$ that $g(x) = ax^2$ for all $x \in \mathbb{R}$. We check this solution:

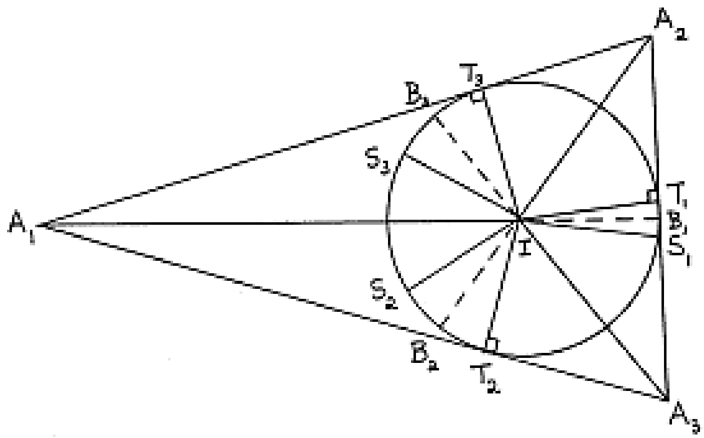
$$\begin{aligned} g(x+y) + g(x-y) &= a(x+y)^2 + a(x-y)^2 \\ &= 2ax^2 + 2ay^2 \\ &= 2g(x) + 2g(y) \end{aligned}$$

and so this is a valid solution for any $a \in \mathbb{R}$.

24. We are required to show that $\triangle M_1M_2M_3 \sim \triangle S_1S_2S_3$. Since $\triangle M_1M_2M_3 \sim \triangle A_1A_2A_3$, with the centroid the point of homothety (are you sure why?), it suffices to show that $\triangle A_1A_2A_3 \sim \triangle S_1S_2S_3$.

Actually, we're telling a white lie here. If two figures are both homothetic to a third, it 'usually' would mean that the first two are homothetic. However, they could be congruent. (This is the only obstacle, and is something unnatural in the definition of homothetic figures.) However, that isn't a problem here: S_1, S_2, S_3 lie on the incircle, while M_1, M_2, M_3 lie on the nine point circle. These circles have different radii because the triangle is not equilateral.

To show that $\Delta A_1A_2A_3 \sim \Delta S_1S_2S_3$, it suffices to show that $S_1S_2 \parallel A_1A_2$, the others will follow by symmetry.



Now $S_1I = S_2I = r$ and $IT_3 \perp A_1A_2$, so it suffices to show that $\angle S_1IT_3 = \angle S_2IT_3$.

$$\begin{aligned}
\angle S_1IT_3 &= \angle S_1B_1 + \angle B_1IT_1 + \angle T_1IT_3 \\
&= 2\angle T_1IB_1 + \angle T_1IT_3 \\
&= 2(\angle A_1IT_2 + \angle T_2IT_1 - 180^\circ) + \angle T_1IT_3 \\
&= 2(\angle A_1IT_2 - \angle A_3) + (180^\circ - \angle A_2) \\
&= \angle T_3IT_2 - 2\angle A_3 + 180^\circ - \angle A_2 \\
&= 360^\circ - \angle A_1 - \angle A_2 - 2\angle A_3 \\
&= \angle A_1 + \angle A_2
\end{aligned}$$

and similarly $\angle S_2IT_3 = \angle A_2 + \angle A_1$.

(IMO 1982 Question 2)

25.

n	$f(n)$
1	1
2	3
3	6
4	2
5	7
6	1
7	8
8	16
9	7
10	17
11	6
12	18
13	5
14	19
15	4
16	20
17	3
18	21
19	2
20	22
21	1
22	23
23	46
24	22
25	47
26	21
27	48
	...

We notice the following two crucial things in the table:-

- The values of n for which $f(n) = 1$ are 1, 6, 21, ... which can be determined as follows: the next term is 3 times the previous plus 3.
- After this occurs, the values of $f(n)$ follow a nice alternating pattern until the next occurrence of a 1.

Let's prove some things.
Let the values of n for which $f(n) = 1$ be listed in order as $b_1, b_2, b_3, \ldots$. Then we see that

$$\begin{aligned}
f(b_n + 2j - 1) &= b_n - j + 3 \qquad (1) \\
f(b_n + 2j) &= 2b_n + j + 3 \qquad (2)
\end{aligned}$$

for every n and for small j. In fact, this pattern holds until $b_n - j + 3$ reaches 1, at which point we have reached b_{n+1} and the pattern starts over again. Now, if $b_n - j + 3 = 1$ then $j = b_n + 2$ and so $b_n + 2j - 1 = 3b_n + 3$. Thus, $b_{n+1} = 3b_n + 3$, as expected.
Thus we have an inductive formula for b_n, but we are going to need a closed formula. We have

$$\begin{aligned}
b_1 &= 1 \\
b_2 &= 3 \cdot 1 + 3 = 3 + 3 \\
b_3 &= 3(3 + 3) + 3 = 3^2 + 3^2 + 3 \\
b_4 &= 3(3^2 + 3^2 + 3) + 3 = 3^3 + 3^3 + 3^2 + 3 \\
&\ldots
\end{aligned}$$

and so we hypothesize that

$$b_n = 3^{n-1} + 3 \cdot \frac{3^{n-1} - 1}{3 - 1} = \frac{5 \cdot 3^{n-1} - 3}{2}$$

which is established by a trivial induction.

We then have

n	b_n
1	1
2	6
3	21
4	66
5	201
6	606
7	1821
8	5466
	...

It is clear from this and the formulae (1) and (2) that we derive 1993 via (1) only and never via (2), the first occurrence being for b_8. To derive 1993, we need $1993 = b_n - j + 3$. that is, $j = b_n - 1990$. Thus

$$b_n + 2j - 1 = b_n + 2(b_n - 1990) - 1 = 3b_n - 3981$$

This does it all. The set of numbers which map to 1993 is $\{3b_n - 3981 : n \geq 8\}$ which is an infinite set; its first member is $3 \cdot b_8 - 3981 = 12417$; and the ratio of the terms approaches 3.

(Proposed at the 1993 IMO)

26. The strategy is to establish a one-to-one correspondence between the set of all permutations without property T and a set of permutations with property T. The proof will be completed by finding some other permutations that satisfy T.

The first thing to note is that given any $a \in \{1, 2, \ldots, 2n\}$ there is a unique $b \in \{1, 2, \ldots, 2n\}$ such that $|a - b| = n$; we will write $a \sim b$.

Let $(y_1, y_2, \ldots, y_{2n})$ be a permutation of the set $1, 2, \ldots, 2n$ $(n \geq 2)$ which does not satisfy property T. Find the number y_r such that $y_r \sim y_1$. Note that by assumption $r > 2$. Now assign the permutation $(y_1, y_2, \ldots, y_{2n})$ to the permutation which has y_1 and y_{r-1} swopped. Note that this permutation has property T implemented by $y_1 \sim y_r$, and it is not implemented at any other point.

Most importantly, note that this assignment is one-to-one: given such a permutation in the range of the assignment, simply find the first element of the pair of numbers that implement property T. Swop that number with the first number listed in the permutation, and we have found the inverse image.

Hence we have a one-to-one assignment from the permutations without property T to a set of permutations that have T exactly once. However, there are more permutations with property T. For example, $(1, n+1, 2, n+2, \ldots, n, 2n)$ has it n times.

Note:
Suppose we followed a similar strategy but tried the following assignment: instead of swopping y_1 to the correct place we swop y_r with y_2. Then indeed we have constructed a map from the set of permutations without T to the set with T, but the map is not one-to-one! For example, in the case $n = 4$, both $(1, 2, 3, 4, 5, 6, 7, 8)$ and $(1, 3, 5, 4, 2, 6, 7, 8)$ are sent to $(1, 5, 3, 4, 2, 6, 7, 8)$. Somehow the information of where the 5 (the y_r) came from is lost. In the correct solution, we know that the swopped entry certainly came from the first position.

A point to ponder. Thanks to Jan van Zyl Smit and Mark Berman for some interesting correspondence on this problem. More than ever lots of experimentation is necessary for solving this problem. The official solution was unfortunately very complicated, recursively developing formulae for the exact sizes of the various sets of interest.

(IMO 1989 Question 6)

27. Assume always, until faced with the inevitable, that there is no monochromatic triangle. Without loss of generality BC is red. We argue according to how many of the lines A_iB are red:-

- at least three lines A_iB, A_jB, A_kB are red. Then A_iC, A_jC, A_kC are blue. Two of A_i, A_j, A_k are not adjacent, say it is A_i and A_j. Then if A_iA_j is coloured red, A_iA_jB is red, while if it is coloured blue, then A_iA_jC is blue.
- at most one line A_iB is red, say it is A_1B. Then A_iB is blue for $2 \leq i \leq 7$ and so A_3A_5, A_5A_7, A_7A_3 are red. Then $A_3A_5A_7$ is red.
- two i have A_iB red. We can suppose these two are 1 and 2, or 1 and 3, or 1 and 4.
  - (a) 1 and 2: A_3B, A_5B, A_7B are all blue, and so A_3A_5, A_5A_7, A_7A_3 are all red. Then $A_3A_5A_7$ is red.
  - (b) 1 and 3: similarly $A_2A_4A_6$ is red.
  - (c) 1 and 4: similarly $A_3A_5A_7$ is red.

(Bulgarian National Olympiad 1993)

28. Put all the primes into k sets, according to the value of the prime modulo k. By the pigeonhole principle one of these sets is infinite; enumerate this set as $x_0 < x_1 < x_2 < \cdots$. Now put $p = x_0$ and $a_n = \frac{x_n - p}{k}$.

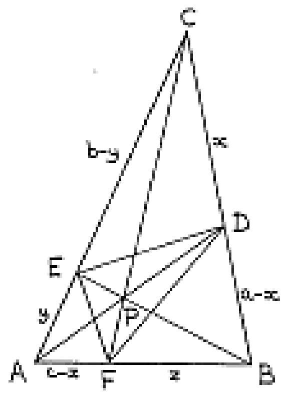
Alternatively, this result can be seen as an immediate application of Dirichlet's theorem.[6]

(Bulgarian IMO selection test 1987)

29. Rather than find P, we are going to structure our thoughts towards finding D, E, F. (Although note that once two of these are determined, the third is.)
Label the diagram as shown and let

$$\alpha = \frac{x}{a},\ \beta = \frac{y}{b},\ \gamma = \frac{z}{c}.$$

(These ratios are of interest because they indicate the relative positions of D, E, F.)



By Ceva's theorem we have $(a-x)(b-y)(c-z) = xyz$, that is, $(1-\alpha)(1-\beta)(1-\gamma) = \alpha\beta\gamma$. Then

$$\begin{aligned} |DEF| &= |ABC| - |CDE| - |AEF| - |BDF| \\ &= |ABC|\ [1 - \alpha(1-\beta) - \beta(1-\gamma) - \gamma(1-\alpha)]. \end{aligned}$$

We want to maximize this, so we want to minimize

$$\alpha(1-\beta) + \beta(1-\gamma) + \gamma(1-\alpha) = \alpha + \beta + \gamma - \alpha\beta - \beta\gamma - \gamma\alpha.$$

But $(1-\alpha)(1-\beta)(1-\gamma) = \alpha\beta\gamma$ and so $1-\alpha-\beta-\gamma+\alpha\beta+\beta\gamma+\gamma\alpha-\alpha\beta\gamma = \alpha\beta\gamma$, in other words, $1 - 2\alpha\beta\gamma = \alpha+\beta+\gamma-\alpha\beta-\beta\gamma-\gamma\alpha$.

[6] Dirichlet's theorem says that every arithmetic progression a, $a+d$, $a+2d$, ... where $d > 0$ and $(a,d) = 1$ contains infinitely many primes. This and related information is mentioned in the South African Mathematical Society Olympiad Training Notes no.2, 'Topics in Number Theory', by Valentin Goranko. The application of Dirichlet's theorem here was noted by Jan van Zyl Smit.

So we want to minimize $1 - 2\alpha\beta\gamma$, that is, maximize $\alpha\beta\gamma$. But

$$\begin{aligned} \alpha\beta\gamma &= \sqrt{\alpha(1-\alpha)\beta(1-\beta)\gamma(1-\gamma)} \\ &\leq \left(\frac{\alpha + (1-\alpha) + \beta + (1-\beta) + \gamma + (1-\gamma)}{6}\right)^{1/3} \\ &= \frac{1}{8} \end{aligned}$$

with equality when $\alpha = \beta = \gamma = \frac{1}{2}$. It follows that P should be chosen to be the centroid.

(Proposed at an IMO)

30. (a) The defining equation suggests that we should solve the equation $\phi^2 = -\phi + 1$. We get $\phi = \frac{\sqrt{5}-1}{2}$. (Of course, only the positive solution, since $a_i \geq 0$.) So we put

$$a_i = \left(\frac{\sqrt{5}-1}{2}\right)^i$$

and it is apparent that this is a solution to the problem.

(b) Suppose b_0, b_1, ... is another sequence having the same properties. Then $a_0 = 1 = b_0$, and if the b_i's are going to differ from the a_i's, they had better do it at b_1 or never at all. In other words, let's put $b_1 = a_1 + c$; it suffices to prove that $c = 0$.

Let's find out what is going on:-

n	a_n	b_n
0	$a_0 = 1$	$b_0 = 1$
1	a_1	$a_1 + c$
2	$a_0 - a_1$	$b_0 - b_1 = a_0 - a_1 - c = a_2 - c$
3	$a_1 - a_2$	$b_1 - b_2 = a_1 + c - (a_2 - c) = a_3 + 2c$
4	$a_2 - a_3$	$b_2 - b_3 = a_2 - c - (a_3 + 2c) = a_4 - 3c$
5	$a_3 - a_4$	$b_3 - b_4 = a_3 + 2c - (a_4 - 3c) = a_5 + 5c$
...		

We hypothesize that for $n \geq 1$, $b_n = a_n + (-1)^{n+1} F_n c$, where F_n is the n^{th} term of the Fibonacci sequence 1, 1, 2, 3, ... No surprises, this is shown by induction. We've already shown that it is true for n up to 5. Suppose it is true up to $n+1$. Then

$$\begin{aligned} b_{n+2} &= b_n - b_{n+1} \\ &= a_n + (-1)^{n+1} F_n c - [a_{n+1} + (-1)^{n+2} F_{n+1} c] \\ &= a_n - a_{n+1} + (-1)^{n+3}[F_n + F_{n+1}] c \\ &= a_{n+2} + (-1)^{n+3} F_{n+2} c \end{aligned}$$

as required.

Assume that $c \neq 0$. Now $a_n \to 0$, since $\phi < 1$, while $F_n c \to \pm\infty$ (depending on whether $c < 0$ or $c > 0$). So eventually we will have some (either the odd or the even) b_n's being less than 0, a contradiction. Thus $c = 0$.

(Australian Mathematics Olympiad 1991)

31. We suppose such a function does exist. If $f(n) = f(m)$ then $n + 1987 = f^2(n) = f^2(m) = m + 1987$, and so f is injective.

We have $f^3(n) = f(f^2(n)) = f(n+1987)$, and also $f^3(n) = f^2(f(n)) = f(n) + 1987$, and so

$$f(n+1987) = f(n) + 1987 \qquad (3)$$

Inductively we can easily show that

$$f(n + 1987q) = f(n) + 1987q \quad (q \geq 0) \qquad (4)$$

This tells us that the function is fully determined by its values on the set $\{0, 1, \ldots, 1986\}$, which we call A.

Suppose $n \in A$. Let $f(n) = 1987q + r$, where $0 \leq r \leq 1986$. Then

$$\begin{aligned} n + 1987 &= f^2(n) \\ &= f(1987q + r) \\ &= 1987q + f(r) \end{aligned}$$

and so $2 \cdot 1987 > n + 1987 = 1987q + f(r) \geq 1987q$, which shows that $q \in \{0, 1\}$. Thus $f(n) \leq 1986 + 1987$ for all $n \in A$.

Now let

$$\begin{aligned} A_1 &= \{n \in A : 0 \leq f(n) \leq 1986\} \\ A_2 &= \{n \in A : 1987 \leq f(n) \leq 1986 + 1987\} \end{aligned}$$

Then $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. Since A has odd cardinality, to derive the required contradiction it now suffices to find a bijection between A_1 and A_2. Okay?

We claim that f itself (or strictly speaking, the restriction of f to A_1) provides a bijection $f : A_1 \to A_2$. Firstly, note that this claim makes sense, because if $n \in A_1$ then $f(n) \in A$ and $f(f(n)) = n + 1987$ which shows that $f(n) \in A_2$. We have already seen that f is injective. To finish it suffices to show that f maps A_1 onto A_2.

Suppose $n \in A_2$. Then $f(n) - 1987 \in A$. By transforming (3) we get that

$$f(m) = f(m - 1987) + 1987$$

for $m \geq 1987$ and since $f(n) \geq 1987$ we get

$$f(f(n) - 1987) = f(f(n)) - 1987 = n$$

Thus $f(n) - 1987 \in A_1$, and it is mapped to n, showing that $f : A_1 \to A_2$ is onto.

(IMO 1987 Question 4)

32. It is clear that we are going to show that the process must terminate. (For otherwise the problem would be solved by an example, which seems implausible.) As in many problems of this type, we want to determine an 'index function' (of five variables) which has certain properties which cannot be maintained indefinitely. More precisely: we find a $\mathbb{N}$-valued function of the values at the five vertices which decreases strictly with each operation. This cannot continue indefinitely, and so the process must terminate.

For the function to be positive valued, we might expect that it would involve absolute values of quantities or squares of quantities. We give one solution of each type.

First solution

As a first guess, we might put

$$f(x, y, z, v, w) = |x| + |y| + |z| + |v| + |w|$$

(This is on the intuition that the procedure is an averaging procedure, and so converts large values into ones closer to 0.) A simple calculation shows that under these circumstances, the function decreases in value by $|x|+|z|-|x+y|-|y+z|$ when the procedure is performed. This might be a negative decrease, so our choice of function is inadequate. It does indicate, however, that we should include the absolute values of the sum of adjacent pairs in our function. Thus, we try

$$\begin{aligned} f(x,y,z,v,w) &= |x|+|y|+|z|+|v|+|w| \\ &+ |x+y|+|y+z|+|z+v|+|v+w|+|w+x| \end{aligned}$$

Then (do it!) the function decreases by $|z+v|+|w+x|-|y+z+v|-|w+x+y|$. For the same reason as before, we now try

$$\begin{aligned} f(x,y,z,v,w) &= |x|+|y|+|z|+|v|+|w| \\ &+ |x+y|+|y+z|+|z+v|+|v+w|+|w+x| \\ &+ |x+y+z|+|y+z+v| \\ &\quad +|z+v+w|+|v+w+x|+|w+x+y| \end{aligned}$$

Then (do it!) the function decreases by $|z+v+w|+|v+w+x|-|z+y+v+w|-|v+w+x+y|$. For the same reason as before, we now try

$$\begin{aligned} & f(x,y,z,v,w) \\ = {} & |x|+|y|+|z|+|v|+|w| \\ + {} & |x+y|+|y+z|+|z+v|+|v+w|+|w+x| \\ + {} & |x+y+z|+|y+z+v| \\ & +|z+v+w|+|v+w+x|+|w+x+y| \\ + {} & |x+y+z+v|+|y+z+v+w| \\ & +|z+v+w+x|+|v+w+x+y|+|w+x+y+z| \end{aligned}$$

Then (do it!) the function decreases by $|v+w+x+z|-|v+w+x+z+2y|$. At last something out of the ordinary has happened. Put $x+y+z+v+w=\Sigma$. So the function decreases by $|\Sigma-y|-|\Sigma+y|$, which is strictly positive, since $\Sigma>0$ and $y<0$. We're done.

Second solution

We derive a quadratic form that has the same properties. So, let the function be

$$f(x_1,x_2,x_3,x_4,x_5)=\sum Q_{ij}x_ix_j$$

where

$$Q_{ij}=\begin{cases} a & \text{if } i=j \\ b & \text{if } |i-j|=1 \\ c & \text{otherwise} \end{cases}$$

This is done on the intuition that the expression should be symmetric in the five variables, and that each point, neighbouring points, and other points should each carry some weights, namely a, b, c. Later we will determine these quantities.

Some extremely miserable calculations (I did them, so you should too) show that under these circumstances the function decreases by $y[(2b-2a-c)(x+y+z)+(c-b)(v+w)]$ when the procedure is performed. A judicious choice of a, b, c will ensure that this quantity is positive. Since $x+y+z+v+w>0$, it now makes sense to try $2b-2a-c=c-b$ and to make it negative, so that the product with y (which is negative) will be positive. For example, we put $a=1$, $b=0$, $c=-1$. We're done.

(IMO 1986 Question 3)

33. Suppose $x_1\le x_2\le\cdots\le x_5$ and $a_1\le a_2\le\cdots\le a_{10}$. Then $\sum_{i=1}^{10}a_i=4\sum_{i=1}^{5}x_i$. Also we have

$$x_4+x_5=a_{10},\quad x_3+x_5=a_9,\quad x_1+x_3=a_2,\quad x_1+x_2=a_1$$

and so

$$\begin{aligned} a_1+a_2+a_9+a_{10} &= 2x_1+2x_3+2x_5+x_2+x_4 \\ &= 2(x_1+x_2+\cdots+x_5)-(x_2+x_4) \\ &= \frac{1}{2}(a_1+a_2+\cdots+a_{10})-(x_2+x_4) \end{aligned}$$

and thus we determine for which i we have $a_i=x_2+x_4$.

Then $x_2 + x_3 + x_4 + x_5 = a_i + a_9$, from which we determine x_1. The rest follow easily.

By the way, this problem admits a generalization as follows: with 5 replaced by any n which is not a power of two.

34. By a result of Gauss[7] it suffices to show that $f(x)$ is irreducible over $\mathbb{Z}[x]$. So suppose $f(x) = g(x)h(x)$ is a non-trivial factorization with $g(x),\ h(x) \in \mathbb{Z}[x]$. Since $g(a_i)\, h(a_i) = f(a_i) = -1$, we have $g(a_i) = \pm 1$, $h(a_i) = \mp 1$. Thus $g(a_i) + h(a_i) = 0$. But then if $g(x) + h(x)$ were a non-zero polynomial, it would be of degree less than n but have the n distinct roots $a_1,\ a_2,\ \ldots,\ a_n$. Thus it must be the case that $g(x) + h(x) = 0$. In other words, $h(x) = -g(x)$, and so $f(x) = -g(x)^2$. This contradicts the fact that $f(x)$ has leading coefficient 1.

(Nordic Mathematical Contest 1992)

35. (i) We see that $f(x + a) \geq \frac{1}{2}$ for all x, and so $f(x) \geq \frac{1}{2}$ for all x. Now

$$\begin{aligned}(f(x+a))^2 &= \frac{1}{4} + f(x) - (f(x))^2 + \sqrt{f(x) - (f(x))^2} \\ &= f(x) - (f(x))^2 - \frac{1}{4} + f(x+a).\end{aligned}$$

Therefore

$$\begin{aligned}f(x+2a) &= \frac{1}{2} + \sqrt{f(x+a) - (f(x+a))^2} \\ &= \frac{1}{2} + \sqrt{(f(x))^2 - f(x) + \frac{1}{4}} \\ &= \frac{1}{2} + \left| f(x) - \frac{1}{2} \right| \\ &= f(x)\end{aligned}$$

from our very first comment. Thus f has period $2a$.

(ii) $f(x) = \frac{1}{2}\left|\sin \frac{\pi}{2}x\right| + \frac{1}{2}$ and $f(x) = \begin{cases} \frac{1}{2} & \text{if } \lfloor x \rfloor \text{ is even} \\ 1 & \text{if } \lfloor x \rfloor \text{ is odd} \end{cases}$ are examples.

[7] This result, occasionally amusingly called Gauss' lemma, states that if $f(x) \in \mathbb{Z}[x]$ is reducible over $\mathbb{Q}[x]$ then it is already reducible over $\mathbb{Z}[x]$.

(IMO 1968 Question 5)

36. Let the colours be named 1, 2, $\ldots$, $p+1$. Choose and fix any vertex of K_n. Let the colour i be represented x_i-many times at this vertex. Then of course $x_1 + x_2 + \cdots + x_{p+1} = n - 1$.

We are going to prove that $x_i \leq p - 1$ for $1 \leq i \leq p + 1$. Suppose for a contradiction that $x_i \geq p$. Let's call colour i red. From the given vertex, then, there are p red edges. By hypothesis the edges connecting the vertices at the other end of these edges must also be red. Thus we get a red copy of K_{p+1}.

Now take a vertex not already featured. (It must exist, because the graph is coloured in more than just red.) There are $p+1$ edges from this vertex to the vertices of the red copy of K_{p+1}. No two of these edges can be coloured with the same colour, else we would have a dichromatic triangle. In particular, red must be used at least once. But then no other colour could be used, for then again we would have a dichromatic triangle. Thus only red is used.

Since the vertex not already featured was arbitrary, the whole graph is coloured red. This is a contradiction. Thus $x_i \leq p - 1$. Hence $n - 1 = x_1 + x_2 + \cdots + x_{p+1} \leq (p+1)(p-1) = p^2 - 1$ showing that $n \leq p^2$.

We now construct a graph with p^2 vertices having the required property. Let the vertices be labelled (i, j) where $1 \leq i,\ j \leq p$. The edge connecting the vertex (i, j) to the vertex (i', j') will be coloured with the colour $p + 1$ if $j = j'$. If $j \neq j'$, then $j - j'$ has a multiplicative inverse in $\{1,\ 2,\ \ldots,\ p-1\}$ modulo p. (Here, for the one and only time, we use the fact that p is prime. In fancy terminology, $\{0,\ 1,\ \ldots,\ p-1\}$ is a *field* when we work modulo p.) Thus, if $j \neq j'$, we colour the edge with colour $(i - i')(j - j')^{-1}$.

It is easy (do it!) to verify that this colouring gives the required colouring of the graph.

(From the contest section of Matematika # 6, 1990)

37.

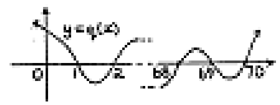
$$\sum_{k=1}^{70} \frac{k}{x-k} - \frac{5}{4} = \frac{\sum_{k=1}^{70} k \prod_{\substack{j=1 \\ j \neq k}}^{70} (x-j)}{\prod_{j=1}^{70} (x-j)} - \frac{5}{4}$$

$$= \frac{4\sum_{k=1}^{70} k \prod_{\substack{j=1\\ j\neq k}}^{70}(x-j) \quad - \quad 5\prod_{j=1}^{70}(x-j)}{4\prod_{j=1}^{70}(x-j)}$$

$$= \frac{p(x)}{q(x)}$$

Let us examine the graphs of $p(x)$ and $q(x)$.

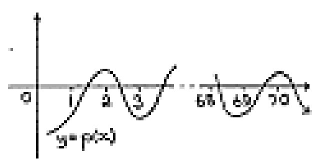
$q(x)$ is easy. It is of degree 70, and has the zeros $1, 2, \ldots, 70$. It has leading coefficient 4.



$p(x)$ is also of degree 70, and has leading coefficient -5. Of course we cannot determine its zeros, but we can get an idea of their location. To see this, note that

$$\begin{aligned}
p(70) &= 4\cdot 70\cdot(70-1)(70-2)\cdots(70-69) > 0\\
p(69) &= 4\cdot 69\cdot(69-1)(69-2)\cdots(69-68)(69-70) < 0\\
\vdots\ &\ \vdots\quad \vdots\\
p(3) &= 4\cdot 3\cdot(3-1)(3-2)(3-4)\cdots(3-69)(3-70) > 0\\
p(2) &= 4\cdot 2\cdot(2-1)(2-3)\cdots(2-69)(2-70) > 0\\
p(1) &= 4\cdot 1\cdot(1-2)(1-3)\cdots(2-69)(1-70) < 0
\end{aligned}$$

We now use the intermediate value theorem to determine the approximate location of 69 of the zeros. Since $p(x)$ has negative leading coefficient, there is another zero which is > 70. Thus we get the graph of $p(x)$.



Now $\frac{p(x)}{q(x)} \geq 0$ if and only if $p(x)$, $q(x)$ have the same sign, and this occurs in the shaded intervals in the diagram; in which $p(x)$ and $q(x)$ have been superimposed.

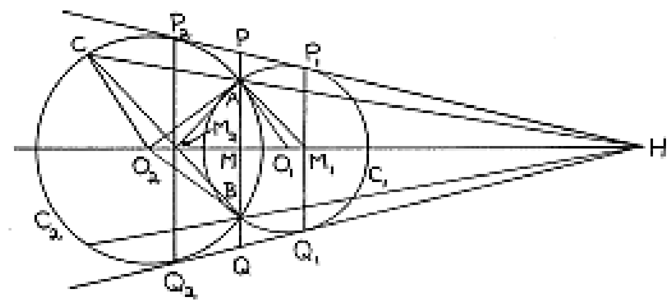
For convenience let us denote $1+2+\cdots+70$ by S. The total lengths of the shaded intervals is the sum of their right hand endpoints minus the sum of their left hand endpoints, which is the sum of the roots of $p(x)$ minus S. The degree 69 term of $p(x)$ is $4S+5S = 9S$, while the degree 70 term is -5. Hence the sum of the roots of $p(x)$ is $-\frac{9S}{-5}$.

Thus the required sum of the lengths is $-\frac{9S}{-5} - S = \frac{4}{5}S = \frac{4}{5}\frac{70\cdot 71}{2} = 1988$.

(IMO 1988 Question 4)

38. We may suppose the circles are of different radii, for otherwise the problem is trivial. Then P_2P_1 and Q_2Q_1 intersect at a point H; the circles are homothetic and H is the point of homothety. Furthermore, H, M_1, O_1, M_2, O_2 are collinear.

Let the other point of intersection of the two circles be B. Let the point of intersection of AB with P_1P_2 be P; with Q_1Q_2 be Q; and with M_1M_2 be M. Extend HA to meet C_2 again in C.



Now A and C correspond under the homothetism and therefore $\angle M_1AO_1 = \angle M_2CO_2$. It thus suffices to prove that $\angle M_2AO_2 = \angle M_2CO_2$. Since $\angle M_2AO_2 = \angle M_2BO_2$ it suffices for this to show that C, M_2, B are collinear (for then ΔO_2BC is isosceles and $\angle M_2BO_2 = \angle M_2CO_2$).

To show that C, M_2, B are collinear we observe that $CM_2 \parallel AM_1$ from the homothetism; hence it suffices to show that $BM_2 \parallel AM_1$.

This is easy: PQ is the radical axis[8] of the two circles and so $P_2P = PP_1$. Hence $M_2M = MM_1$, and so $\angle AM_1M = \angle AM_2M$. Therefore $\angle AM_1M = MM_2B$, and so $BM_2 \parallel AM_1$ by equal alternate angles.

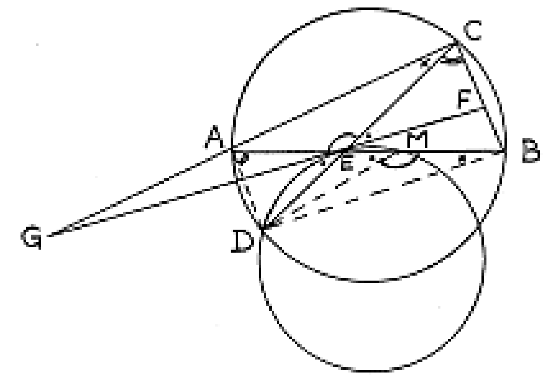
(IMO 1983 Question 2)

[8] Recall that the power of a point X with respect to a circle is the product of the distances of X to any two points, lying on the circle, which are collinear with X. That this definition makes sense is an easy consequence of similar triangle ratios. By definition, the radical axis of two circles is the set of points which have equal powers with respect to each circle. In the case that circles intersect then it is the line of intersection.

8.4 Solutions to the Rhodes Tests

Solutions: Test I

1. Join D to A, M and B.



Note that by the alternate segment theorem, $\angle CEF = \angle GED = \angle EMD$. Also $\angle DAM = \angle DCB$, and hence $\Delta CEF \,|||\, \Delta AMD$. Also, since $\angle ECG = \angle MBD$ and $\angle CEG = 180° - \angle CEF = 180° - \angle EMD = \angle DMB$ we get that $\Delta CGE \,|||\, \Delta BDM$.

Hence $\frac{EF}{EC} = \frac{MD}{MA}$, $\frac{EG}{EC} = \frac{MD}{MB}$. Therefore $\frac{EG}{EF} = \frac{MA}{MB} = \frac{t}{1-t}$.

(IMO 1990 Question 1)

2. Let $A = a - 1$, $B = b - 1$, $C = c - 1$. Then

$$\begin{aligned}
q &:= \frac{abc - 1}{(a-1)(b-1)(c-1)} \\
&= \frac{(A+1)(B+1)(C+1) - 1}{ABC} \\
&= \frac{ABC + AB + AC + BC + A + B + C}{ABC} \\
&= 1 + \frac{1}{C} + \frac{1}{B} + \frac{1}{A} + \frac{1}{BC} + \frac{1}{AC} + \frac{1}{AB}
\end{aligned}$$

Now $A \geq 1,\ B \geq 2,\ C \geq 3$ and hence

$$1 < q \leq 1 + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} + \frac{1}{6} + \frac{1}{3} + \frac{1}{2} < 4$$

and so $q = 2$ or $q = 3$. Also, if $A \geq 3,\ B \geq 4,\ C \geq 5$ then

$$q \leq 1 + \frac{1}{5} + \frac{1}{4} + \frac{1}{3} + \frac{1}{20} + \frac{1}{15} + \frac{1}{12} = 1 + \frac{59}{60} < 2$$

a contradiction. Hence $A < 3$ and so $a = 2$ or $a = 3$. The remainder of the problem is merely testing of cases.

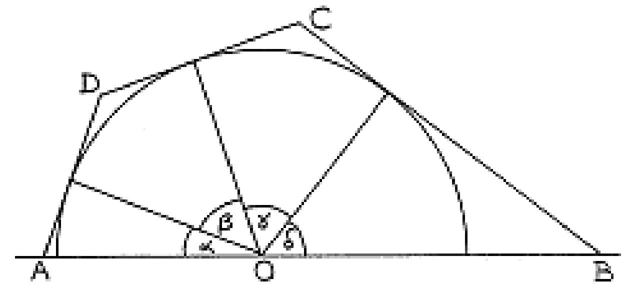
- $a = 2,\ q = 2$: then $2 = \frac{2bc-1}{(b-1)(c-1)}$, so $2(b-1)(c-1) = 2bc-1$, a contradiction modulo 2. So there are no solutions here.
- $a = 3,\ q = 3$: then $3 = \frac{3bc-1}{2(b-1)(c-1)}$, so $6(b-1)(c-1) = 3bc-1$, a contradiction modulo 3. So there are no solutions here.
- $a = 2,\ q = 3$: then $3 = \frac{2bc-1}{(b-1)(c-1)}$, so $3(b-1)(c-1) = 2bc-1$. Expanding and rearranging we get $(b-3)(c-3) = 5$. Thus $b - 3 = 1,\ c - 3 = 5$, giving $b = 4,\ c = 8$.
- $a = 3,\ q = 2$: then $2 = \frac{3bc-1}{2(b-1)(c-1)}$, so $4(b-1)(c-1) = 3bc-1$. Expanding and rearranging we get $(b-4)(c-4) = 11$. Thus $b - 4 = 1,\ c - 4 = 11$, giving $b = 5,\ c = 15$.

(IMO 1992 Question 1)

Solutions: Test II

1. We label the diagram as shown, and suppose the circle has radius 1. Then

$$AD + BC = \left(\tan\alpha + \tan\frac{\beta}{2}\right) + \left(\tan\frac{\gamma}{2} + \tan\delta\right).$$

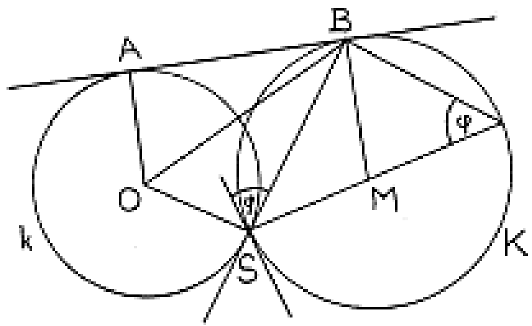


We know that angle β equals the angle at B because they are both supplementary to the angle at D, and therefore β and δ are complementary. Likewise α and γ are complementary. Using this, together with the half-angle formula for tangent (easily obtained from the double-angle formula), the last expression becomes

$$\begin{aligned} AD + BC &= \cot\gamma + \frac{1-\cos\beta}{\sin\beta} + \frac{1-\cos\gamma}{\sin\gamma} + \cot\beta \\ &= \frac{1}{\sin\beta} + \frac{1}{\sin\gamma} \\ &= \sec\delta + \sec\alpha \\ &= OB + OA \\ &= AB \end{aligned}$$

(IMO 1985 Question 1)

2. Let the common tangent contact k at A, and let the centres of k and K be O and M respectively. Extend SM to cut K at P.



$\angle SBM = \angle BSM = 90° - \phi$, and so $\angle ABS = \phi$ since $\angle ABM = 90°$. Since BA and BS are tangents to k from a common point, $SB = AB$ and $\angle ABO = \angle SBO = \frac{\phi}{2}$. Therefore, by considering $\triangle OBA$ we have that $\tan\frac{\phi}{2} = \frac{r}{AB} = \frac{r}{SB}$.

Now from the tangent-chord theorem, $\angle BPS = \phi$, and so by considering $\triangle SPB$ we have $\sin\phi = \frac{SB}{2R}$. Hence $\frac{r}{R} = 2 \cdot \frac{r}{SB} \cdot \frac{SB}{2R} = 2\tan\frac{\phi}{2}\sin\phi$.

Thus it suffices to note that $2\tan\frac{\phi}{2}\sin\phi = \left(2\sin\frac{\phi}{2}\right)^2$, which follows from the fact that $\sin\phi = 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}$.

(Australian Mathematics Olympiad 1985 Question 5)

Solutions: Test III

1. $(1,1)$ is one such pair. It suffices to show that given one such pair (x,y) we can construct another such pair (x',y') with $x' \geq x$ and $y' > y$. The existence of infinitely many distinct solution pairs then follows inductively.

So suppose (x,y) is a solution pair; w.l.o.g. $x \leq y$. There exists $y' \in \mathbb{N}$ such that $y^2 + m = xy'$. Note that $y' > y$, for otherwise

$$y^2 < y^2 + m = xy' \leq yy' \leq y^2$$

a contradiction. We claim that (y,y') is the required pair (i.e. we put $x' = y$).

Suppose p is a prime so that $p|y$ and $p|y'$. Then $p|y^2 + m$ and so $p|m$. Since $p|x^2 + m$, we get that $p|x^2$. Hence $p|x$. Thus $p|y$ and $p|x$, and so we get the required contradiction, and deduce that y and y' are relatively prime.

By construction $y'|y^2 + m$.

We need to show that $y|y'^2 + m$. But since x and y are relatively prime, it suffices to show that $y|x^2\left(y'^2 + m\right)$. Now

$$\begin{aligned} x^2\left(y'^2 + m\right) &= (xy')^2 + mx^2 \\ &= (y^2 + m)^2 + mx^2 \\ &= y^4 + 2my^2 + m^2 + mx^2 \\ &= y^4 + 2my^2 + m(x^2 + m) \end{aligned}$$

which y does indeed divide.

(Proposed at the 1992 IMO)[3]

2. $\frac{A'I}{AI} = \frac{A'B}{AB}$ since BI bisects $\angle B$. (Use the sine rule.) Similarly $\frac{A'I}{AI} = \frac{A'C}{AC}$. Hence

$$\frac{A'I}{AI} = \frac{A'B + A'C}{AB + AC} = \frac{a}{c+b}$$

[3] This question is curiously similar to question B2 of the 1995 South African Mathematical Olympiad, which went as follows: 'Find all pairs (m,n) of natural numbers with $m < n$ such that n divides $m^2 + 1$ and m divides $n^2 + 1$.'

So

$$\frac{AI}{AA'} = \frac{AI}{AI + IA'} = \frac{1}{1 + \frac{A'I}{AI}} = \frac{1}{1 + \frac{a}{c+b}} = \frac{c+b}{a+b+c}$$

Similar expressions hold for the other ratios, and it follows that we are required to prove that

$$\frac{1}{4} < \frac{(a+b)(b+c)(c+a)}{(a+b+c)^3} \le \frac{8}{27}.$$

From the Arithmetic-Geometric mean inequality we have that

$$\begin{aligned} & \sqrt[3]{\frac{a+b}{a+b+c} \cdot \frac{b+c}{a+b+c} \cdot \frac{c+a}{a+b+c}} \\ \le \; & \frac{1}{3}\left[\frac{a+b}{a+b+c} + \frac{b+c}{a+b+c} + \frac{c+a}{a+b+c}\right] \\ = \; & \frac{2}{3} \end{aligned}$$

and so

$$\frac{(a+b)(b+c)(c+a)}{(a+b+c)^3} \le \frac{8}{27}.$$

For the other inequality, we use the transformation $a = x + y$, $b = y + z$, $c = z + x$[10] and from this it easily follows that we must show

$$(y + 2x + z)(z + 2y + x)(x + 2z + y) > 2(x + y + z)^3.$$

Now this is apparently true, since we can consider the following factorisation of the left hand side:-

$$\begin{aligned} & (y + 2x + z)(z + 2y + x)(x + 2z + y) \\ = \; & [(y + x + z) + x] \cdot [(z + y + x) + y] \cdot [(x + z + y) + z] \\ = \; & 2(y + x + z)^3 + \cdots \\ > \; & 2(x + y + z)^3 \end{aligned}$$

(IMO 1991 Question 1)

[10] x, y and z can be found as the distances of the vertices of the triangle to the points of tangency of the incircle, equivalently, $x = s - a$, $y = s - b$, $z = s - c$ where $s = \frac{a+b+c}{2}$ is the semiperimeter.

Solutions: Test IV

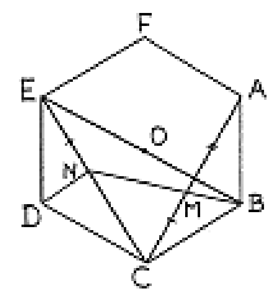
1. Fix $p \ge 1$. For $0 \le r \le a_p$, let B_r be those of $a_1, a_2, \ldots$ which are $\equiv_{a_p} r$. By the pigeonhole principle we have that there exists an r such that B_r is an infinite set.

   Suppose a_q is the smallest member of B_r which exceeds a_p. Then for all a_m in B_r with $m > q$, we have that $a_m \equiv_{a_p} a_q$, and so $a_m = xa_p + a_q$ for some $x \in \mathbb{N}$. So put $y = 1$, and the problem is solved.

   (IMO 1975 Question 2)

2. **First solution**

   For the record, we note that $\angle ECB = \angle DCA = 90°$ and $\angle DEN = \angle DCE = \angle BCA = \angle CAB = 30°$.

   We have $\triangle DEN \equiv \triangle BCM$ (since $NE = CM$) and therefore $\angle END = \angle CMB$ and $\angle EDN = \angle CBM$.

   Hence

$$\begin{aligned} \angle BND &= \angle BNC + \angle CND \\ &= (90° - \angle CBM) + (180° - \angle END) \\ &= 270° - (\angle CBM + \angle END) \\ &= 270° - (\angle CBM + \angle CMB) \\ &= 270° - 150° \\ &= 120°. \end{aligned}$$

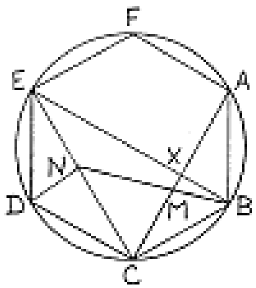
Clearly also $\angle BOD = 120°$. So DB subtends the same angle at N as O. Thus N lies on the arc DOB i.e. on the circle centred at C and radius CO.

Hence

$$\frac{AC}{AM} = \frac{CE}{CN} = \frac{CE}{CB} = \tan\angle EBC = \tan 60° = \sqrt{3}.$$

Second solution

Suppose that the sides of the hexagon are of length 1. Then $AC = \sqrt{3}$. Let X be the point of intersection of AC and EB. We now apply Menelaus' theorem to $\triangle CEX$ and the line BMN:



$$\frac{CN}{NE} \cdot \frac{EB}{BX} \cdot \frac{XM}{MC} = -1.$$

Let $\frac{AC}{AM} = \alpha$. We calculate all of the ratios appearing here in terms of α, and hence solve for α.

Firstly, $EB = 2$ and $BX = -\frac{1}{2}$. Also, by elementary ratio and proportion, $\frac{CN}{NE} = \frac{1}{\alpha-1}$. Likewise $MC = \sqrt{3}\left(1 - \frac{1}{\alpha}\right)$. Hence $XM = XC - MC = \frac{\sqrt{3}}{2} - MC = \sqrt{3}\left(\frac{1}{\alpha} - \frac{1}{2}\right)$. Thus

$$\begin{aligned} \frac{1}{\alpha-1} \cdot \frac{2}{-1/2} \cdot \frac{\sqrt{3}\left(\frac{1}{\alpha} - \frac{1}{2}\right)}{\sqrt{3}\left(1 - \frac{1}{\alpha}\right)} &= -1 \\ \Rightarrow \frac{1}{\alpha-1} \cdot 2 \cdot \frac{2-\alpha}{\alpha-1} &= 1 \\ \Rightarrow 4 - 2\alpha &= (\alpha-1)^2 \\ \Rightarrow 0 &= \alpha^2 - 3 \\ \Rightarrow \sqrt{3} &= \alpha \end{aligned}$$

(IMO 1982 Question 5)

3. The first few members of the orbit of 1 under f are
1, 2, 3, 4, 6, 5, 8, 12, 10, 7, 16, 24, 20, 14, 9.

This suggests writing the orbit as

1				
2	3			
4	6	5		
8	12	10	7	
16	24	20	14	9

where we understand that the action of f is as one reads a piece of writing: scanning from left to right and then from top to bottom. Note that the odd numbers appear on the diagonal, and otherwise each number is twice the number appearing immediately above it.

We need to prove this. Let the rows be numbered 1, 2, 3, ... We claim that the k^{th} term in the n^{th} row is $(2k-1)\,2^{n-k}$, for $1 \leq k \leq n$. The proof is by induction. This is clear from our experiments for $1 \leq n \leq 5$. Suppose it is true for all entries up to the n^{th} row. The last entry in the n^{th} row is thus $2n-1$.

Now $f(2n-1) = 2^n$ and so the claim is true for the first entry in the $(n+1)^{th}$ row. Also

$$\begin{aligned} f\left((2k-1)2^{n-k}\right) &= \frac{(2k-1)2^{n-k}}{2} + 2^{n-k} \\ &= (2k+1)\,2^{n-k-1} \\ &= (2(k+1)-1)\,2^{n-(k+1)} \end{aligned}$$

as required to complete the induction step.

Suppose now $x = \beta 2^\alpha \in \mathbb{N}$ with β odd. We solve in integers the equation $(2k-1)2^{n-k} = x = \beta 2^\alpha$. Thus

$$2k - 1 = \beta, \quad n - k = \alpha$$

and so

$$k = \frac{\beta+1}{2}, \quad n = \alpha + k = \alpha + \frac{\beta+1}{2}.$$

Thus demonstrates that x occurs in the $\frac{\beta+1}{2}^{th}$ position in the $\alpha + \frac{\beta+1}{2}^{th}$ row, and this determination is unique (more or less via unique factorisations).

In particular if $x = 1992 = 249 \cdot 2^3$ then $k = 125$ and $n = 128$. So 1992 is the 125^{th} term in the 128^{th} row, in other words, it is the

$$1 + 2 + \cdots + 126 + 127 + 125 = 8253^{rd}$$

term.

(Proposed at the 1992 IMO)

Solutions: Test V

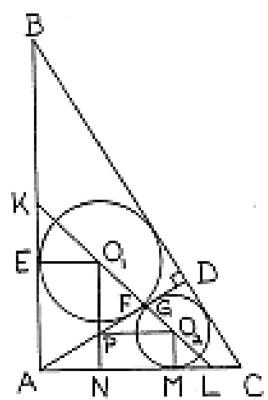
1. Start at some vertex v_0. Imagine yourself walking along distinct edges of the graph, numbering them 1, 2, ... as you encounter them, until you cannot go any further without reusing an edge.

   If there are edges which are not numbered, one of them has a vertex which has been visited, since $\mathcal{G}$ is connected. Starting with this vertex, continue to walk along unused edges, resuming the numbering where you left off, until once again you can go no further. Repeat this procedure until all the edges are numbered.

   We now prove that the numbering satisfies the stated condition that at each vertex belonging to two or more edges the GCD of the numbers of the edges meeting at that vertex is 1. Let v be such a vertex. If $v = v_0$ i.e. v is the starting point, then one of the edges meeting at v is labelled one, and so the GCD at v is 1. If $v \neq v_0$, suppose the first time you encountered v on the walk was at the end of the edge labelled r. At that time there were one or more unused edges at v, one of which was then labelled $r+1$. The GCD of any set containing both r and $r+1$ is 1.

   (IMO 1991 Question 4)

2. As usual denote AB by c, AC by b, BC by a; and also AD by h. Consider the circles C_1 and C_2 inscribed in triangles ABD and ADC and denote their respective centres by O_1 and O_2. Let the respective radii be r_1 and r_2. Let E and F be the points of contact of C_1 with AB and AD respectively, and G and M the points of contact of C_2 with AD and AC respectively. Let O_1N be perpendicular to AC and O_2P perpendicular to NO_1. Then



   - $O_1N = EA = AF = h - r_1$;
   - $O_1P = O_1N - PN = O_1N - O_2M = h - r_1 - r_2$;
   - $O_2P = MN = AM - AN = AG - r_1 = h - r_2 - r_1$.

   Hence $PO_1 = PO_2$ and thus $\angle O_1O_2P = 45°$. Thus $\angle O_2LM = 45°$ and thus $ML = O_2M = r_2$. Consequently

   $$AL = AM + ML = AG + r_2 = h - r_2 + r_2 = h.$$

   Similarly $AK = h$. Hence

   $$\frac{S}{T} = \frac{ah}{h^2} = \frac{a}{h} = \frac{a^2}{ah} = \frac{a^2}{bc} = \frac{b^2+c^2}{bc} \geq 2.$$

   (IMO 1988 Question 5)

3. We construct a set T containing 2047 integers, all less than 100000, which satisfy the conditions.

   The set consists of all positive integers whose base 3 representation have at most 11 digits, each of which is either 0 or 1. There are $2^{11} - 1 = 2047 > 1983$ such numbers, and the largest is

   $$11111111111 = 1 + 1\cdot 3 + 1\cdot 3^2 + \cdots + 1\cdot 3^{10} = 88573 < 100000.$$

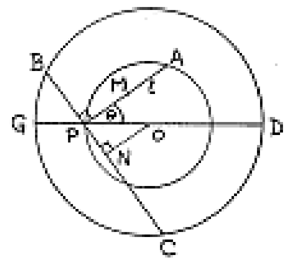
   Now suppose $x + z = 2y$ for some $x,\ y,\ z \in T$. The number $2y$ consists only of the digits 0 and 2. Hence x and z must match digit for digit, and so $x = y = z$. Hence T contains no arithmetic progressions of length 3.

   (IMO 1983 Question 5)

8.5 Solutions to the Wits Tests

Solutions: Wits Camp June 1995: Test I

1. (i) Let $\angle OPA = \theta$, let M be the midpoint of PA and let N be the midpoint of BC.



Then

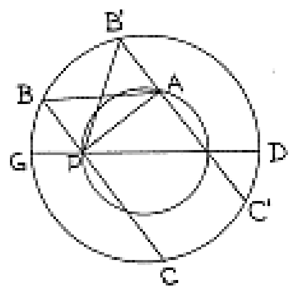
$$\begin{aligned} & BC^2 + CA^2 + AB^2; \\ = & (BP + PC)^2 + PC^2 + PA^2 + BP^2 + PA^2; \\ = & 2[BP^2 + PA^2 + PC^2 + BP \cdot PC]. \end{aligned}$$

Now

$$\begin{aligned} PA &= 2r\cos\theta \\ BP &= BN - PN = \sqrt{R^2 - r^2\cos^2\theta} - r\cos\theta \\ PC &= NC + PN = \sqrt{R^2 - r^2\cos^2\theta} + r\cos\theta \\ BP \cdot PC &= GP \cdot PD = (R-r)(R+r) = R^2 - r^2 \end{aligned}$$

Hence we get that $BC^2 + CA^2 + AB^2 = 6R^2 + 2r^2$, a constant.

(ii) Draw in the parallel to BC that passes through A, and let it meet the larger circle at B' and C'. Then the midpoint of BA is also the midpoint of PB'. Since B is arbitrary, so is B'. Hence the locus is the midpoints of the line segment PB', where B' varies arbitrarily over the larger circle. This is clearly the circle with radius $R/2$ and centre at the midpoint of PO.



(IMO 1988 Question 1)

2. Firstly, $(a+b)^7 - a^7 - b^7 = 7ab(a+b)(a^2+ab+b^2)^2$. Hence we equivalently need that $7 \nmid ab(a+b)$, $7^3 | a^2+ab+b^2$.

First method: lucky guess
$(a+b)^2 > a^2+ab+b^2 \geq 7^3 = 343$. Hence $a+b \geq \lfloor\sqrt{343}\rfloor = 18$. So we try $a = 1$, $b = 18$, it works, we're finished.

Second method: constructive
$a^3 - b^3 = (a-b)(a^2+ab+b^2)$; thus we have $a^3 \equiv_{7^3} b^3$.

Now $\phi(7^3) = 6 \cdot 7^2 = 3 \cdot 98$. Hence $(2^{98})^3 = 2^{3\cdot 98} \equiv_{7^3} 1$. So we try $a = 1$, $b = 2^{98}$. Since $2^{98} \equiv_7 2^2 = 4$, we have $a \pm b \not\equiv_7 0$, and so the required conditions are indeed satisfied.

(IMO 1984 Question 2)

3. Let d denote the usual Euclidean distance. As we travel along the path, choose the first point which comes within 1/2 distance of a vertex of the square. Name this point P_1, and the vertex V_1. Travel further until we 'approach' (for the first time) one of the two vertices adjacent to V_1. Name this vertex V_2 and an appropriate point P_2. We now name V_3 and V_4 in the usual cyclic manner. We name points P_3 and P_4 correspondingly, although note that P_3 may occur on the path before P_2 does.

Let $[L < P_2]$ denote that part of the path which occurs before P_2, and similarly define $[L > P_2]$. Let us examine the edge V_1V_4. Let

$$\begin{aligned} A &= \{x \in V_1V_4 : d(x, [L < P_2]) \leq 1/2\} \\ B &= \{x \in V_1V_4 : d(x, [L > P_2]) \leq 1/2\} \end{aligned}$$

Note that from the hypothesis we have that $A \cup B = V_1V_4$. Furthermore, $A \cap B \neq \emptyset$. (The essential reason for this is that both A and B are topologically closed sets. At any rate, it is clear from the conditions that A and B consist of the union of closed intervals and possibly finitely many discrete points. Clearly two such sets cannot cover an interval disjointly.)

Choose $z \in A \cap B$. Choose $X \in [L < P_2]$ such that $d(X, z) \leq 1/2$; likewise choose $Y \in [L > P_2]$. Then $d(X, Y) \leq 1$ and the path length between X and Y is at least 198, since P_2 is intermediate.

(IMO 1982 Question 6)

Solutions: Wits Camp June 1995: Test II

1. Our strategy is as follows: to show $\angle BFP = \frac{1}{2}\angle B$, it suffices to show that $\sin\angle BFP = \sin\left(\frac{1}{2}\angle B\right)$, since both angles are acute. To achieve this we aim to apply the sine rule in $\triangle FBP$. Thus we first determine the lengths of FB, BP, FP.

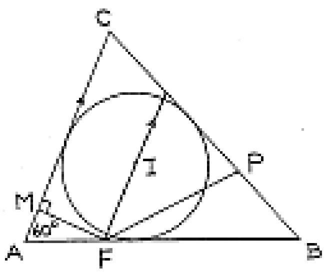
   We may suppose that $BC = 3$ and so $BP = 1$. Then $R = \frac{3}{2\sin 60^\circ} = \sqrt{3}$, and thus

$$\begin{aligned} r &= 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \\ &= 4\sqrt{3}\,\frac{1}{2}\,\sin\frac{B}{2}\sin\frac{C}{2} \\ &= 2\sqrt{3}\,\sin\frac{B}{2}\sin\frac{C}{2} \end{aligned}$$

   This suggests putting $\angle B = 60^\circ + 2x$, $\angle C = 60^\circ - 2x$. Thus

$$\begin{aligned} r &= 2\sqrt{3}\sin(30^\circ + x)\sin(30^\circ - x) \\ &= \sqrt{3}[\cos 2x - \cos 60^\circ] \\ &= \sqrt{3}\left[\cos 2x - \frac{1}{2}\right] \end{aligned}$$

   Let M be the orthogonal projection of F onto AC, then $FM = r$ and hence



$$AF = r\cdot\frac{2}{\sqrt{3}} = 2\left[\cos 2x - \frac{1}{2}\right]$$

   Also
$$\frac{AB}{\sin\angle C} = \frac{3}{\sin 60^\circ},$$
   and so
$$AB = 2\sqrt{3}\sin(60^\circ - 2x).$$

   Hence

$$\begin{aligned} FB &= AB - AF \\ &= 2\sqrt{3}\sin(60^\circ - 2x) - 2\cos 2x + 1 \\ &= 2\sqrt{3}\,\frac{\sqrt{3}}{2}\,\cos 2x - 2\sqrt{3}\,\frac{1}{2}\,\sin 2x - 2\cos 2x + 1 \\ &= \cos 2x - \sqrt{3}\sin 2x + 1 \\ &= 2\cos^2 x - 2\sqrt{3}\sin x\cos x \\ &= 2\cos x[\cos x - \sqrt{3}\sin x] \\ &= 4\cos x\left[\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x\right] \\ &= 4\cos x\cos(60^\circ + x) \end{aligned}$$

By applying the cosine rule, we get

$$\begin{aligned} FP^2 &= [4\cos x\cos(60^\circ + x)]^2 + 1^2 \\ &\quad -2\cdot 4\cos x\cos(60^\circ + x)\cdot 1\cdot\cos(60^\circ + 2x) \\ &= 1 + 8\cos x\cos(60^\circ + x)\cdot \\ &\quad [2\cos x\cos(60^\circ + x) - \cos(60^\circ + 2x)] \\ &= 1 + 8\cos x\cos(60^\circ + x)\cos 60^\circ \\ &= 1 + 4\cos x\cos(60^\circ + x) \\ &= 1 + 2[\cos(60^\circ + 2x) + \cos 60^\circ] \\ &= 2 + 2\cos(60^\circ + 2x) \\ &= 2[1 + \cos(60^\circ + 2x)] \\ &= 4\cos^2(30^\circ + x) \end{aligned}$$

and so
$$FP = 2\cos(30^\circ + x)$$

Finally
$$\frac{\sin\angle B}{FP} = \frac{\sin\angle PFB}{PB},$$

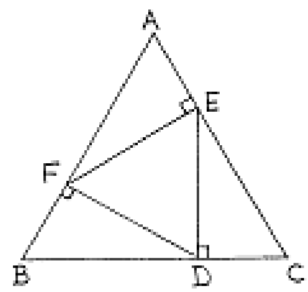
and so
$$\frac{\sin(60^\circ + 2x)}{2\cos(30^\circ + x)} = \frac{\sin\angle PFB}{1},$$

Thus
$$\sin(30^\circ + x) = \sin \angle PFB$$
as required.

(Proposed at the 1991 IMO)

2. Suppose S is such a partition. Colour the elements of the two subsets red and blue respectively, and suppose that every right angled triangle is dichromatic.

Trisect the edges of $\triangle ABC$ to form points D, E, F as shown. Without loss of generality F and E are red.
Then all of AC, except E, is blue. Hence D is red.
By repeating this argument we will see that all points except D, E, F are blue. This is an apparent contradiction.



(IMO 1983 Question 4)

3. We are being asked to show that for every such sequence we have $\sum_{i=0}^{\infty} \frac{{x_i}^2}{x_{i+1}} \geq 4$, and that 4 is best here.

Let L be the best quantity. We can choose a sequence so that $\sum_{i=0}^{\infty} \frac{{x_i}^2}{x_{i+1}} \approx L$. (It is just a little more than L.) Now put $y_i = \frac{x_{i+1}}{x_1}$.
Then $1 = y_0 \geq y_1 \geq \cdots$, and so
$$\begin{aligned} L &\approx \sum_{i=0}^{\infty} \frac{{x_i}^2}{x_{i+1}} \\ &= \frac{1}{x_1} + x_1 \sum_{i=0}^{\infty} \frac{{y_i}^2}{y_{i+1}} \\ &\geq \frac{1}{x_1} + x_1 L \end{aligned}$$

Hence $L \geq \frac{1}{x_1} + x_1 L$, and so by the arithmetic-geometric mean inequality we get $L \geq 2\sqrt{L}$. Thus $L^2 \geq 4L$, and so $L \geq 4$, since $L > 0$.

To show that 4 is best, we can take $x_i = \frac{1}{2^i}$.

(IMO 1982 Question 3)

Solutions: Wits Camp July 1995: Test III

1. By symmetry we have

$$\sum_{i<j} x_i x_j (x_i + x_j) = \frac{1}{2} \sum_{i \neq j} x_i x_j (x_i + x_j) = \sum_{i \neq j} x_i{}^2 x_j$$

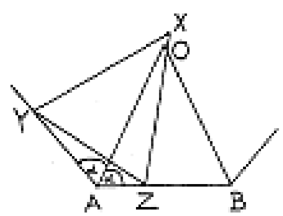
We may suppose that $x_1 \geq x_2 \geq \cdots \geq x_n$. Put $y_1 = x_1$, $y_2 = x_2$, $\ldots$, $y_{n-2} = x_{n-2}$, $y_{n-1} = x_{n-1} + x_n$. Then

$$\begin{aligned} \sum_{i \neq j} y_i{}^2 y_j - \sum_{i \neq j} x_i{}^2 x_j &= (y_{n-1}{}^2 - x_{n-1}{}^2 - x_n{}^2) \sum_{j=1}^{n-1} x_j \\ &= 2x_{n-1}x_n \sum_{j=1}^{n-1} x_j \\ &\geq 0 \end{aligned}$$

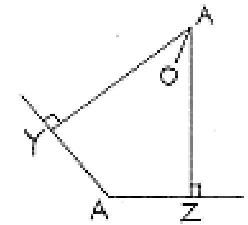
This demonstrates that among sets containing at least three members, we may for the purposes of maximisation reduce the argument to sets containing one member less. It follows that the problem is reduced to maximising the expression $x_1 x_2 (x_1 + x_2)$ (where $n = 2$). Easily this maximum is $\frac{1}{4}$.

(Proposed at the 1991 IMO)

2. Let $\angle OAB = \alpha$. $\angle X = \angle O = 180° - 2\alpha$, and so $XYAZ$ is a cyclic quadrilateral. Since $\angle OAZ = \alpha$, we thus have that A, O, X are collinear.

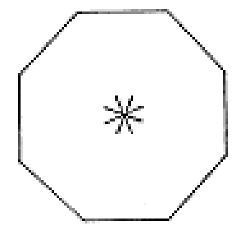


Hence X lies on the line AO. Clearly then, as YZ moves from AB through to CA, X traces out the shaded line segment. The extreme value of X, call it A', corresponds to the situation where $AZ = AY$, and then $\angle XZA = 90°$.



We may suppose for convenience that $|AB| = 1$. Then

$$\begin{aligned} |AO| &= \frac{1}{2} \csc \frac{\pi}{n} \\ |AA'| &= \csc \frac{2\pi}{n} \end{aligned}$$



Thus X traces out a line segment of length $\csc \frac{2\pi}{n} - \frac{1}{2} \csc \frac{\pi}{n}$ emanating from O and pointing away from A. The complete locus then is an asterisk consisting of n such segments, one for each choice of A.

(IMO 1986 Question 4)

3. Suppose $a + d = 2^m$ and $b + c = 2^n$. We want to find out which power is larger, so we analyse $a + d - b - c$. This has the same sign as $a^2 + ad - ab - ac = a^2 + bc - ab - ac = (a - c)(a - b) > 0$, and so $a + d > b + c$, and $m > n$.

Now $b(2^n - b) = bc = ad = a(2^m - a)$. Therefore

$$2^n(b - a2^{m-n}) = b2^n - a2^m = b^2 - a^2 = (b - a)(b + a)$$

and so $2^n | (b - a)(b + a)$. But if $b \pm a \equiv_4 0$ we get $b \equiv_2 0$, a contradiction. Hence we either have $2^{n-1} | b - a$ or $2^{n-1} | b + a$; call the one that 2^{n-1} does divide x. Then $0 < x \leq b + a < b + c = 2^n$, and so $x = 2^{n-1}$.

Now if y is a common prime factor of a and b, then y is a common prime factor of $b \pm a$, and so $y = 2$. Thus $(a, b) = 1$.

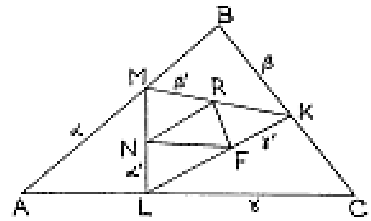
Also $2^{n-1} = 2^n - 2^{n-1} = b + c - x = c \pm a$, and so in the same manner as above we get $(a, c) = 1$.

Hence $(a, bc) = 1$. But $a | bc$, and so $a = 1$.

(IMO 1984 Question 6)

Solutions: Wits Camp June 1995: Test IV

1. Let $|AM| = \alpha|AB|$,
$|BK| = \beta|BC|$,
$|CL| = \gamma|CA|$,
$|LN| = \alpha'|LM|$,
$|MR| = \beta'|MK|$,
$|KF| = \alpha'|KL|$.
Then



$$\begin{aligned}
|AMR| &= |ABC| \cdot \beta\alpha\beta' \\
|CKR| &= |ABC| \cdot (1-\beta)(1-\alpha)(1-\beta') \\
|BKF| &= |ABC| \cdot \gamma\beta\gamma' \\
|ALF| &= |ABC| \cdot (1-\gamma)(1-\beta)(1-\gamma') \\
|CLN| &= |ABC| \cdot \alpha\gamma\alpha' \\
|BMN| &= |ABC| \cdot (1-\alpha)(1-\gamma)(1-\alpha')
\end{aligned}$$

Hence

$$\begin{aligned}
|AMR| \cdot |CKR| &= \beta(1-\beta)\alpha(1-\alpha)\beta'(1-\beta') \cdot |ABC|^2 \\
&\leq \left(\frac{1}{2}\right)^6 |ABC|^2
\end{aligned}$$

by the arithmetic-geometric mean inequality. Similarly for the other two corresponding products; hence

$$\begin{aligned}
&\sqrt[6]{|AMR| \cdot |CKR| \cdot |BFK| \cdot |ALF| \cdot |CLN| \cdot |BMN|} \\
\leq\ &\sqrt[6]{\left(\frac{1}{2}\right)^{18} |ABC|^6} \\
=\ &\left(\frac{1}{2}\right)^3 |ABC| \\
=\ &\frac{1}{8}|ABC|
\end{aligned}$$

(Proposed at the 1988 IMO)

2. Each of the members of $\mathcal{M}$ have 9 possible prime factors. For $\mathcal{M} \ni x = 2^{\alpha_2}3^{\alpha_3}\cdots 23^{\alpha_{23}}$ we consider the map

$$f : \mathcal{M} \to \{0,1\}^9 : x \to (\alpha_2,\ \alpha_3,\ \ldots \alpha_{23})$$

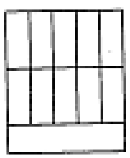
where entries are reduced modulo 2. If we choose any $2^9 + 1 = 513$ elements of $\mathcal{M}$, then there are two, call them x and y, such that $f(x) = f(y)$. Thus xy is a perfect square.

We can repeat this, removing a pair from $\mathcal{M}$ whose product is a square each time, continuing until there are only 511 members of $\mathcal{M}$ left. Thus we can remove 737 pairs x_i, y_i with the property that $x_i y_i = z_i{}^2$ for some $z_i \in \mathbb{N}$.

Since $737 > 513$, an argument similar to the previous one shows that there exist i and j such that $z_i z_j$ is a perfect square. Then $x_i y_i x_j y_j$ is a fourth power.

(IMO 1985 Question 4)

3. The second player can ensure that the maximum sum is ≤ 6. Tile the board with dominoes (one irregular) as indicated.



Clearly the second player can ensure that no 1×2 domino will contain 2 1's. Since any 3×3 square includes three 1×2 dominos, the second player can ensure that three zeros appear inside any 3×3 square, as required.

The first player can ensure that a maximum sum of 6 is attained. Label the board like a chessboard, with rows $\{1,\ 2,\ \ldots,\ 5\}$ and columns $\{a,\ b,\ \ldots,\ e\}$. Put a 1 in $c3$. By symmetry we may suppose a 0 is marked in the 4^{th} or 5^{th} row. Now put a 1 in $c2$.

If permitted, a 1 will be marked in $c1$ on the next move. At that point, either $\{a,\ b\}$ or $\{d,\ e\}$ will be empty in rows $\{1,\ 2,\ 3\}$. Say it is $\{a,\ b\}$. Then the first player can get three 1's in $\{a,\ b\} \times \{1,\ 2,\ 3\}$, and then $\{a,\ b,\ c\} \times \{1,\ 2,\ 3\}$ will have six 1's.

So we suppose a 0 is marked in $c1$. Then mark a 1 in $b3$. We have the following situation:

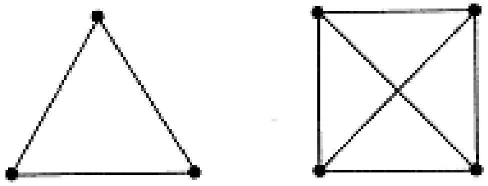
		0		
		1		
	1	1		

The first player now follows a waiting strategy: if the second player marks in $\{a,\ d\} \times \{1,\ 2,\ 3\}$ then so does he; if the second player marks in $\{b\} \times \{1,\ 2\}$ then so does he. Then either $\{a,\ b,\ c\} \times \{1,\ 2,\ 3\}$ or $\{b,\ c,\ d\} \times \{1,\ 2,\ 3\}$ will have six 1's, as required.

(Proposed at the 1994 IMO)

Solutions: Wits Camp June 1995: Test V

1. In the diagram nine edges are used.



We show that it is impossible to fulfil the requirements with eight edges. Suppose it is.

Since $\frac{7 \cdot 3}{2} > 8$, there exists a vertex v_1 with degree ≤ 2. Hence there exist $v_2, \ldots, v_5$ that are not joined to v_1 by an edge. By hypothesis we have a copy of K_4, which uses up 6 vertices. There are thus only 2 vertices left to be used. However, $v_1 v_6 v_2$, $v_1 v_7 v_8$, $v_6 v_7 v_3$ are triangles, each requiring an edge, each necessarily distinct from the other two required edges. In other words, we require three more edges, a contradiction.

(Proposed at the 1989 IMO)

2. Since $\{1, 2, \ldots, n\}$ is a complete residue system modulo n, and $(k, n) = 1$, so is $\{k, 2k, \ldots, nk\}$; and then so is this set reduced modulo n: call it $\{k = m_1, m_2, \ldots, m_n = 0\}$.

Suppose $1 \leq j \leq n$. We examine the two numbers m_j and m_{j+1}.

Note that $m_{j+1} = \begin{cases} m_j + k & \text{if} \quad m_j + k < n \\ m_j + k - n & \text{if} \quad m_j + k \geq n \end{cases}$. Since $1 \leq j < n$ we have $m_{j+1} \neq k$. Therefore:

- If $m_j + k < n$, then m_{j+1} has the same colour as $|m_{j+1} - k| = m_j$;
- If $m_j + k \geq n$, then m_{j+1} has the same colour as $|m_{j+1} - k| = n - m_j$, which in turn has the same colour as m_j.

(IMO 1985 Question 2)

3. Suppose n colours are used. On any given circle, it is possible to have $2^n - 1$ different possible colour combinations. Hence, if we take 2^n circles, each having centre 0 and radius < 1, then two of them (say with radii $0 < r < s < 1$) will have the same colours appearing.

For the problem it suffices to show that there is a point Y on the circle of radius r such that $c(Y)$ is the circle of radius s. For this we need show that for some such Y,

$$r + \frac{a(Y)}{r} = s.$$

But certainly this is satisfied when we take $a(Y) = r(s - r)$. (Note that this is legitimate, since $0 < r(s-r) < 1 < 2\pi$.)

(IMO 1984 Q3)

8.6 Solutions to the Kent Tests

Solutions: Kent Camp July 1995: Test I

1. We first solve the problem with 1989 replaced by $3 \times 117 = 351$ and 17 replaced by 3. A possible solution to this is:-

Set	Members			Sum
1	59	351	118	528
2	58	350	120	528
3	57	349	122	528
⋮	⋮	⋮	⋮	528
57	3	295	230	528
58	2	294	232	528
59	1	293	234	528
60	117	292	119	528
61	116	291	121	528
⋮	⋮	⋮	⋮	528
116	61	236	231	528
117	60	235	233	528

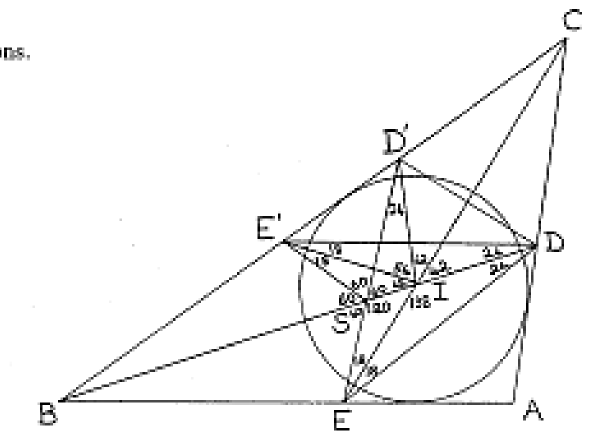
We now need to include a further 14 elements into each A_i. Since there are now evenly many elements to include, we can use some sort of pairing or balancing procedure, and this turns out to be quite elementary. For example: for $3 \times 117 + 1 \leq x \leq 10 \times 117$ we put x into A_x (indices are taken $\equiv_{117}$) and for $10 \times 117 + 1 \leq x \leq 17 \times 117$ we put x in A_{1-x}. This balancing procedure ensures that the required conditions are satisfied.

(IMO 1989 Question 1)

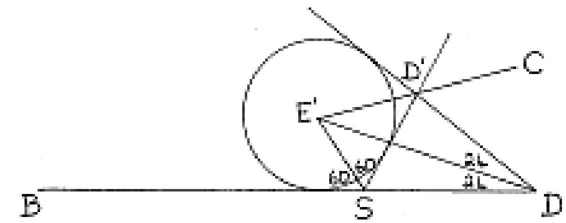
2. $\angle DIE = 180° - 24° - 18° = 138°$. Hence $\angle CIB = 138°$, and $\angle ICB + \angle IBC = 42°$. Therefore $\angle C + \angle B = 84°$, and so $\angle A = 96°$.

Let E' on BC be the image of E through the line of symmetry BD, and let D' on BC be the image of D through the line of symmetry EC. Let S be the point of intersection of ED' and BD.

The angles shown in the diagram are then straightforward consequences of these constructions and other elementary considerations.



Now note that $\angle E'DD' = 90° - \angle CED - \angle E'DE = 24°$. From the next diagram - only relevant information is displayed - we see that E' is one of the excentres of $\triangle SDD'$. Thus $\angle DD'C = \angle SD'E'$.



Hence

$$\begin{aligned}\angle DD'C &= \frac{\angle DD'C + \angle SD'E'}{2} \\ &= \frac{180° - \angle ED'D}{2} \\ &= \frac{\angle D'EC + 90°}{2} \\ &= \frac{18° + 90°}{2} \\ &= 54°\end{aligned}$$

Thus $\angle C = 72°$, and $\angle B = 12°$.

(Proposed at the IMO 1992)

3. Given any horizontal line, and scanning from left to right, we join the first to the second vertex on that line with an edge, then the third to the fourth, etc. If the number of vertices on any line is odd, then the last one will be isolated. We then apply a similar procedure to the vertical lines.

We now have a graph, which in principle could have many components. But the important thing is that each component will consist of either an isolated vertex, a path, or a cycle of even length. (Even because the edges of a cycle alternate in being horizontal and vertical.)

We now colour the vertices of the paths and cycles alternately red and blue, and colour the isolated vertices arbitrarily. Since each line consists of various portions of paths and cycles, and at most one isolated vertex, the difference between the number of red and blue vertices on the line is either 0 or 1.

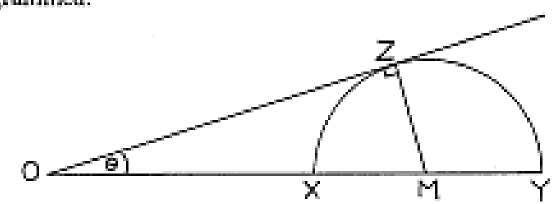
(IMO 1986 Question 6)

Solutions: Kent Camp July 1995: Test II

1. The case where AB and CD are parallel is trivial, so we suppose that this is not the case. Let these lines meet at O, forming an (acute) angle θ.

First note that $AD \parallel BC$ if and only if $\frac{OA}{OB} = \frac{OD}{OC}$. We thus analyse the ratio $\frac{OA}{OB}$.

Consider any two rays at an angle θ and any semicircle with diameter on the one ray and a point of tangency on the other ray, as diagrammed.



Then

$$\frac{OX}{OY} = \frac{OM - MZ}{OM + MZ} = \frac{1 - \frac{MZ}{OM}}{1 + \frac{MZ}{OM}} = \frac{1 - \sin\theta}{1 + \sin\theta}$$

which depends on θ only - call it $K(\theta)$. (The actual value of the constant is irrelevant, and need not be calculated.) Conversely, if $\frac{OX}{OY} = K(\theta)$, then the semicircle is indeed tangent.

Thus $BC \parallel AD$ iff $\frac{OA}{OB} = \frac{OD}{OC}$ iff $\frac{OD}{OC} = K(\theta)$ iff the circle with diameter CD is tangent to AB.

(IMO 1984 Question 4)

2. Number the points $P_1, P_2, \ldots, P_{2n-1}$ in order around the circle. Indices will be considered $\equiv_{2n-1}$. It is clear that a colouring is good iff there exist black P_i and P_j with $i - j \equiv_{2n-1} \in \{n+1, n-2\}$.

If $k \leq n-2$ then there is a bad colouring: let $P_1, P_3, \ldots, P_k$ be black and the rest white.

If $k = n$ then all colourings are good: this is immediate from the pigeonhole principle.

It remains to consider the case $k = n - 1$. We consider two possibilities:

- $3 \nmid 2n - 1$. Then $2n - 1 \equiv_3 \in \{1,\ 2\}$, and so $n \equiv_3 \in \{0,\ 1\}$, and $\{n + 1,\ n - 2\} \equiv_3 \in \{2,\ 1\}$. If we colour the points $P_3,\ P_6,\ \ldots,\ P_{3(n-1)}$ black then for this colouring to be good we need to be able to solve $3x - 3y \equiv_{2n-1} \in \{n + 1,\ n - 2\}$ where $1 \leq x,\ y \leq n - 1$. Putting $z = x - y$ we get $3z \equiv_{2n-1} \in \{n+1,\ n-2\}$ where $1 \leq z \leq n - 2$. Then $3z \in \{n+1,\ n-2\}$. This contradicts $\{n + 1,\ n - 2\} \equiv_3 \in \{2,\ 1\}$. So we have a bad colouring.
- $3|2n-1$. We show that all colourings are good. Divide the set of points into three disjoint sets depending on their value $\equiv_3$. Suppose there is a bad colouring. Then for each black point there is a unique corresponding white point, and it belongs to the same one of the three sets (since $n + 1 \equiv_3 0$). Since each set has oddly many elements, there is at least one more white than black in each set, and so there are at most $n - 2$ blacks altogether, a contradiction.

(IMO 1990 Question 2)

3. **Solution of Mark Berman:**
If there are more than 1994 cards, the pigeonhole principle tells us immediately that at any stage there must be a child with two cards, so the game cannot end.

If there are exactly 1994 cards, then colour the children alternately blue and pink (no sexist undertones intended). Suppose that initially all 1994 cards are held by a pink child. Any move adds or subtracts two from the number of cards held by the set of pink children, and similarly for the blue children. Hence at any stage each set has an even number of cards. Now the game only ends if each child holds one card, which means that the pink children hold 997 cards and the blue children hold 997 cards, a contradiction.

Lastly we consider the case of there being less than 1994 cards. Let's suppose that the children are all shy and will initially not speak to each other. We call a particular card a 'relationship

opener' of two adjacent children if it is the first card passed between them.

Now when a card is responsible for opening a relationship, let us suppose that the two children involved attach some sentimental value to it. We may suppose at each move that the child passing two cards will always prefer to pass a card with sentimental value to the appropriate neighbour with whom they share that sentiment. Then the situation never arises where the child is in the dilemma of wanting to pass the same relationship opening card to both neighbours (since each card opens at most one relationship). In other words, once a card has become a relationship opener it remains loyal to the two relevant children for the rest of the game.

In the group there are 1994 pairs of consecutive children, but at any stage there are less than 1994 existing relationship openers, since there are less than 1994 cards in total. Hence at least one relationship never gets opened. These two children can only receive cards, and so at some stage they have to stop participating (since there are only finitely many cards). When this happens, the neighbours on either side of them are now in exactly the same position as the original two had been in. In this way, a wave of apathy spreads around the circle until no-one is participating any longer. Then surely the game has ended.

(Proposed at the 1994 IMO)

Solutions: Kent Camp July 1995: Test III

1. First we recall the in-out formula $\frac{n}{k}\binom{n-1}{k-1} = \binom{n}{k}$. In fact, it would be more precise to write $n\binom{n-1}{k-1} = k\binom{n}{k}$, since this makes sense even for $k = 0$.

   Returning to the problem: first note that

$$\begin{aligned} p_n(k) &= \binom{n}{k} p_{n-k}(0) \\ \sum_{k=0}^{n} p_n(k) &= n! \end{aligned}$$

   Hence

$$\begin{aligned} \sum_{k=0}^{n} k \cdot p_n(k) &= \sum_{k=0}^{n} k \cdot \binom{n}{k} p_{n-k}(0) \\ &= n \sum_{k=0}^{n} \binom{n-1}{k-1} p_{n-k}(0) \\ &= n \sum_{k=0}^{n-1} \binom{n-1}{k} p_{n-k-1}(0) \\ &= n \sum_{k=0}^{n-1} p_{n-1}(k) \\ &= n(n-1)! \\ &= n! \end{aligned}$$

   (IMO 1987 Question 1)

2. By substituting $wz + a$ for z we get

$$f(wz + a) + f(w^2 z + wa + a) = g(wz + a)$$

   By substituting $w^2 z + wa + a$ for z we get

$$f(w^2 z + wa + a) + f(w^3 z + w^2 a + wa + a) = g(w^2 z + wa + a)$$

   But $w^3 = 1$ and $w^2 + w + 1 = 0$, and so

$$f(w^2 z + wa + a) + f(z) = g(w^2 z + wa + a)$$

   Hence

$$\begin{aligned} f(z) &= g(z) - f(wz + a) \\ &= g(z) - g(wz + a) + f(w^2 z + wa + a) \\ &= g(z) - g(wz + a) + g(w^2 z + wa + a) - f(z) \end{aligned}$$

   and so

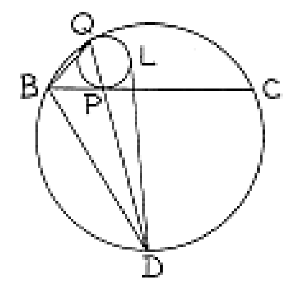
$$f(z) = \frac{1}{2}\left[g(z) - g(wz + a) + g(w^2 z + wa + a)\right]$$

   (Proposed at the 1989 IMO)

3. First let D be the midpoint of the complementary arc BC. We show that DW is tangent to both circles (and hence A, W, D are collinear).

   Method 1

   Consider only one of the inscribed circles, and let L be the point of tangency to the circle from D. Let P and Q be the points of contact of this circle with the chord BC and the arc BC respectively.



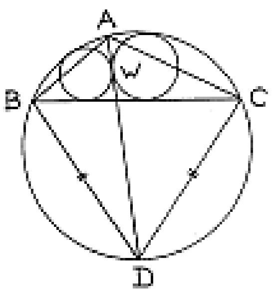
   The two circles are homothetic with Q the point of homothety. P and D correspond, and hence Q, P, D are collinear. Now $\angle BQD = \angle PBD$ since $BD = DC$, and considering $\angle BDQ$ to be a common angle, we get that $\triangle DBP \,|||\, \triangle DQB$. Therefore $\frac{DB}{DP} = \frac{DQ}{DB}$ and so $DB^2 = DP \cdot DQ$. But it is a well known result that $DP \cdot DQ = DL^2$, and so $DB = DL$.

   Now consider the other inscribed circle, and let L' be the point of tangency to the circle from D. As above we get $DC = DL$, and so $DL = DL'$. Hence $L = L' = W$.

Method 2

Consider inversion in the circle with centre D and radius $DB = DC$. This inversion maps the arc BC to the chord BC and vice versa. Both circles are invariant, hence W stays fixed and DW is tangent to both circles.

In particular, since W stays fixed, it lies on the circle of inversion. Thus $DB = DW$.



In summary: we have shown AWD is a straight line. It is now clear that $\angle BAW = \angle CAW$.
Now $\angle DBW = \angle DWB$ since $DB = DW$. Thus

$$\begin{aligned} \angle ABW &= \angle DWB - \angle BAD \\ &= \angle DBW - \angle CBD \\ &= \angle CBW \end{aligned}$$

This W is the incentre of ΔABC.

(Proposed at the 1992 IMO)

Solutions: Kent Camp July 1995: Test IV

1. We have $g(x) = f(x)^2 - 9 = (f(x)+3)(f(x)-3)$. Suppose for a contradiction $g(x)$ has at least 1996 distinct integer solutions. Let $x_1 < x_2 < \cdots < x_n$ be the distinct integer solutions of $f(x)+3$ and $y_1 < y_2 < \cdots < y_m$ be the distinct integer solutions of $f(x)-3$.

   If $x_i = y_j$ then $f(x_i)+3 = 0 = f(y_j)-3$, and so $6 = f(y_j) - f(x_i)$, a contradiction. Thus the x_i's are different from the y_j's. Thus $n + m \geq 1996$. Now since $n,\ m \leq 1995$, we have that $n,\ m \geq 1$.

   Suppose $f(x) = \sum_{k=0}^{1995} a_k x^k$. Then for any i and j we get

   $$\sum_{k=0}^{1995} a_k {x_i}^k = -3, \quad \sum_{k=0}^{1995} a_k {y_j}^k = 3$$

   and so by subtraction, we get $\sum_{k=1}^{1995} a_k \left({y_j}^k - {x_i}^k\right) = 6$.

   Therefore $(y_j - x_i)|6$. Since $n \geq 1$, it now follows that $m \leq 9$. Likewise $n \leq 9$. So $1995 < n + m \leq 18$, a contradiction.

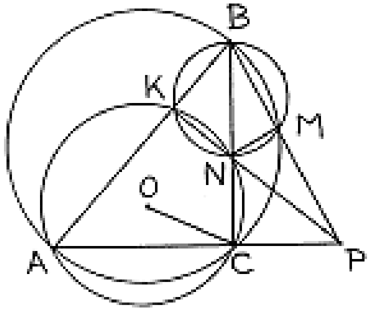
   (An improvement of a proposal at the 1991 IMO, further improvements on the size of the various degrees is possible.)

2. The centres of the three circles are not collinear, since the centre O and the centre of the circumcircle of ΔABC lie on the perpendicular bisector of AC, but the centre of the circumcircle of ΔKBN does not. Hence the three radical axes of the circles (namely AC, KN, BM) coincide at the radical centre which we call P. [11]

   Note that $BMNK$ is cyclic, and the power of P with respect to this quadrilateral's circumcircle is $PM \cdot PB = PN \cdot PK$. But the latter is the power of P with respect to the circle with centre O,

[11] If three circles intersect in pairs then the three radical axes intersect at a common point, called the radical centre, if and only if the centres of the circles are not collinear. If the centres are collinear then the three radical axes are parallel.

and hence equals $PO^2 - OC^2$.[12]



Similarly, $PMNC$ is easily seen to be cyclic, and the power of B with respect to this quadrilateral's circumcircle is $BM \cdot BP = BN \cdot BC$. Again, this is the power of B with respect to the circle with centre O, and thus is equal to $BO^2 - OC^2$.

By subtraction we get $BO^2 - PO^2 = BP(BM - PM) = (BM + PM)(BM - PM) = BM^2 - PM^2$. Now

$$\begin{aligned}\cos\angle NMP &= \frac{MP^2 + OM^2 - OP^2}{2 \cdot MP \cdot MO} \\ &= \frac{BM^2 + OM^2 - BO^2}{2 \cdot MP \cdot MO} \\ &= \cos\angle NMB \cdot \frac{MP}{BM}\end{aligned}$$

Thus $\cos\angle NMP$ and $\cos\angle NMB$ have the same sign, and are therefore both 0. Thus $\angle NMB = 90°$.

(IMO 1985 Question 5)

3. (a) Consider the binary number $\epsilon = d_n d_{n-1} \cdots d_2 d_1$ where $d_i = 0$ if card i is correctly placed and 1 if not. (Note that card '0' is invisible in this index.) Then $\epsilon = 0$ iff the game has finished, and $0 < \epsilon < 2^n - 1$ if not. Whenever a move is made, it is clear that

[12] If a circle has centre O and radius r then the power of X can conveniently be expressed as the difference of squares $(OX + r)(OX - r)$.

ϵ decreases, since $d_l 0 \cdots 0 > 0 d_{l-1} \cdots d_1$. Hence the game lasts at most $2^n - 1$ moves.

(b) For the game to last $2^n - 1$ moves, ϵ must start at $2^n - 1$ and must decrease by 1 at each move. Imagine, rather, that the game is played backwards. Then ϵ starts at $0 \cdots 000$, then becomes $0 \cdots 001$, then $0 \cdots 010$, etc. Clearly, then, the card in the rightmost position with a zero index must jump (into the 'invisible' zeroth position). This is deterministic, and so there is at most one game lasting $2^n - 1$ moves. (This does not prove that such a game exists, since we haven't determined if such a sequence is legally possible.)

(c) Let the cards be arranged as $0, n, n-1, \ldots, 2, 1$. The card 0 does not move until card n reaches position 0. Thus we may delete card number 0 and can n can be renamed 0 (since, until it reaches 0, it is never the card to be picked up). Inductively this reduced game lasts $2^{n-1} - 1$ moves, at which point we reintroduce 0, giving the following configuration: $0, n-1, \cdots, 3, 2, 1, n$. After one more move, we have $n, 0, n-1, \cdots, 3, 2, 1$. Now again ignoring the fixed card n, the game lasts a further $2^{n-1} - 1$ moves, again by the induction hypothesis. Hence the game lasts $2^{n-1} - 1 + 1 + 2^{n-1} - 1 = 2^n - 1$ moves.

(Proposed at the 1994 IMO)

Solutions: Kent Camp July 1995: Test V

1. Let $x = 5^{25}$. Then

$$\begin{aligned}
& \frac{5^{125}-1}{5^{25}-1} \\
= & \frac{x^5-1}{x-1} \\
= & 1+x+x^2+x^3+x^4 \\
= & (x^2+3x+1)^2 - 5x(x+1)^2 \\
= & (x^2+3x+1)^2 - \left[5^{13}(x+1)\right]^2 \\
= & \left[x^2+3x+1-5^{13}(x+1)\right]\ \left[x^2+3x+1+5^{13}(x+1)\right]
\end{aligned}$$

and each factor is greater than 1. (Each is of order 5^{50}.)

(Proposed at the IMO 1992)

2. First recall that $\cos 2\theta = 2\cos^2\theta - 1$ and $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. So putting $x = \cos(\pi a)$, we get

$$\begin{aligned}
0 &= \cos(3\pi a) + 2\cos(2\pi a) \\
&= 4\cos^3(\pi a) - 3\cos(\pi a) + 4\cos^2(\pi a) - 2 \\
&= 4x^3 + 4x^2 - 3x - 2 \\
&= (2x+1)(2x^2+x-2)
\end{aligned}$$

and so $x \in \left\{-\frac{1}{2}, \frac{-1\pm\sqrt{17}}{4}\right\}$. Clearly $x \neq \frac{-1-\sqrt{17}}{4}$. If $x = -\frac{1}{2}$ we easily get $a = \frac{2}{3}$, since $0 < a < 1$.

It thus now suffices to show that $\cos(\pi a) = \frac{\sqrt{17}-1}{4}$ is impossible, and surely we will use the fact that $a \in \mathbb{Q}$. Recall in fact that $a \in \mathbb{Q}$ implies that the set $\{\cos(n\pi a) : n \in \mathbb{Z}\}$ is finite. We aim to show then that this set is infinite.

We claim inductively that $\cos(2^n\pi a) = \frac{a_n+b_n\sqrt{17}}{4}$, where a_n, b_n are odd integers, for all $n \geq 0$. We have $a_0 = -1$, $b_0 = 1$. Inductively we get

$$\begin{aligned}
\cos(2^{n+1}\pi a) &= 2\cos^2(2^n\pi a) - 1
\end{aligned}$$

$$\begin{aligned}
&= 2\left(\frac{a_n+b_n\sqrt{17}}{4}\right)^2 - 1 \\
&= \frac{\frac{1}{2}{a_n}^2 + \frac{17}{2}{b_n}^2 - 4 + a_nb_n\sqrt{17}}{4}
\end{aligned}$$

and so

$$\begin{aligned}
a_{n+1} &= \frac{1}{2}{a_n}^2 + \frac{17}{2}{b_n}^2 - 4 \\
b_{n+1} &= a_nb_n
\end{aligned}$$

which are odd, as required. Now notice that

$$a_{n+1} = \frac{1}{2}{a_n}^2 + \frac{17}{2}{b_n}^2 - 4 \geq \frac{1}{2}{a_n}^2 + \frac{17}{2} - 4 = \frac{1}{2}({a_n}^2+9) > a_n$$

It follows that $\{\cos(2^n\pi a) : n \geq 0\}$ forms an infinite set, contradicting the rationality of a.

(Proposed at the 1991 IMO)

8.7 Solutions to the IMO

1. XY is the radical axis of the two circles, and P lies on this axis. Hence $PC \cdot PM = PB \cdot PN$, and so M, N, C, B are cocyclic. (Note that we don't say that $MNCB$ is a cyclic quadrilateral. The order of the letters depends on whether P lies on the line segment XY or merely on the line XY. So we stick with the more conservative statement.)

   Suppose AM and DN intersect at W. To show that W lies on XY, we need to show that W has the same power with respect to each circle, and for this it suffices to show that A, D, N, M are cocyclic. This comes from

$$\begin{aligned} \angle MND + \angle DAM &\equiv \angle MNB + \angle BND + \angle DAM \\ &\equiv \angle MCB + \angle AMC + \angle DAM \\ &\equiv \angle MCA + \angle AMC + \angle CAM \\ &\equiv 0 \end{aligned}$$

   Note:
   Here all the angles are directed angles, and so we refuse to draw a diagram. A diagram based proof requires two cases: where P is exterior and where P is interior to the line segment XY.

2. Consider the Cauchy-Schwarz inequality in the form

$$\sum x \sum y \geq \left(\sum \sqrt{xy}\right)^2.$$

   Using this form, we have that $[a(b+c)+b(a+c)+c(a+b)] \cdot \left[\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)}\right] \geq \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$.

   Now $ab = \frac{1}{c}$, for example, and so rewriting, we get

$$\begin{aligned} & 2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \cdot \left[\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)}\right] \\ \geq\ & \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2. \end{aligned}$$

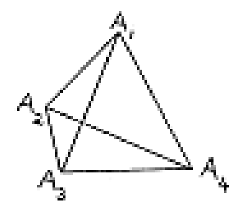
   Thus

$$\begin{aligned} & \left[\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)}\right] \\ \geq\ & \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \\ \geq\ & \frac{1}{2} \cdot 3\sqrt[3]{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}} \\ =\ & \frac{3}{2} \end{aligned}$$

3. We claim that $n = 4$ is the only integer satisfying the conditions of the problem. For $n = 4$, let $A_1A_2A_3A_4$ be a unit square and let $p_1 = p_2 = p_3 = p_4 = \frac{1}{6}$. All we need to do now is show that there is no solution for $n = 5$, which implies there is no solution for $n \geq 5$.

   Suppose there is a solution for $n = 5$. Let us write $[ijk]$ for $|\triangle A_iA_jA_k|$, which then equals $r_i + r_j + r_k$. We are going to use two lemmas:

   Lemma 1 *Suppose $A_1A_2A_3A_4$ is a convex quadrilateral. Then $r_1 + r_3 = r_2 + r_4$.*

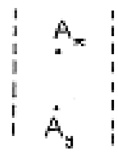


   Proof: $[123] + [143] = [214] + [234]$. ∎

   Lemma 2 *Suppose $r_1 = r_2$. Then for $x,\ y \in \{3,\ 4,\ 5\}$ we have that A_1 and A_2 are equidistant to the line A_xA_y.*

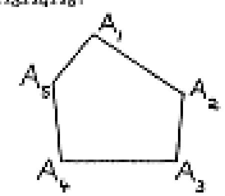
   Proof: Note that $[abc]$ is equal to $\frac{1}{2}|bc|$ times the perpendicular distance of a to the line determined by b and c. Since $[1xy] = [2xy]$ we get that the perpendicular distances from A_1 and A_2 to the line A_xA_y are the same. ∎

In the notation of the lemma, A_1 and A_2 must lie somewhere in the dotted-line set.

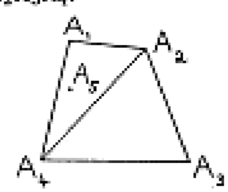


We now complete the proof by considering three cases:

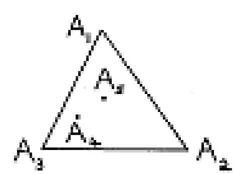
- The convex hull is a pentagon $A_1A_2A_3A_4A_5$: $r_1 + r_3 = r_2 + r_4 = r_5 + r_3$ and so $r_1 = r_5$. By permutation of coefficients we get that $r_1 = r_2 = r_3 = r_4 = r_5$. Since A_2, A_3, A_4 are all on the same side of the line A_1A_5, we get from Lemma 2 that A_2, A_3, A_4 are collinear, a contradiction.



- The convex hull is a quadrilateral $A_1A_2A_3A_4$: Suppose $A_5 \in \Delta A_1A_2A_4$. Then $r_5 + r_3 = r_2 + r_4 = r_1 + r_3$ and so $r_1 = r_5$. Now A_1 and A_5 are on the same side of A_2A_4, and so from Lemma 2 we get that $A_1 = A_5$, a contradiction. Similarly if $A_5 \in \Delta A_2A_3A_4$ we deduce that $A_5 = A_3$, a contradiction.



- The convex hull is a triangle $A_1A_2A_3$: (In this case the indices are not necessarily ordered.) Since both A_4 and A_5 occur in the interior of $\Delta A_1A_2A_3$, we have that $[123] = [124] + [234] + [314] = [125] + [235] + [315]$. Thus $r_4 = r_5$. From Lemma 2 we get that $A_4A_5 \parallel A_1A_2$, $A_4A_5 \parallel A_2A_3$, and $A_4A_5 \parallel A_3A_1$, a contradiction.



4. **Based on the solution of David Hatton:**

$$
\begin{aligned}
x_{i-1} + \frac{2}{x_{i-1}} &= 2x_i + \frac{1}{x_i} \\
\Rightarrow 2x_i^2 - \left(x_{i-1} + \frac{2}{x_{i-1}}\right) x_i + 1 &= 0 \\
\Rightarrow (2x_i - x_{i-1})\left(x_i - \frac{1}{x_{i-1}}\right) &= 0
\end{aligned}
$$

and so $x_i = \frac{x_{i-1}}{2}$ or $x_i = \frac{1}{x_{i-1}}$. Thus we can think of this problem as a game, starting with x_0 and then performing 1995 moves, each of which is either a halving (H) or an inversion (V) of the number previously arrived at.

Before proceeding, we establish certain relationships concerning H and V. (These can be called 'commutation relationships'.) Let us also consider a doubling move D. We have:-

$$
\begin{aligned}
V^2 &= I \\
VH &= DV \\
VD &= HV \\
D^{-1} &= H
\end{aligned}
$$

We claim that after n moves, the game can be transcribed as $D^\alpha V^\beta$, where $\beta \in \{0,1\}$ and $|\alpha| + \beta \leq n$. This is a trivial inductive consequence of $VD^\alpha V^\beta = D^{-\alpha}V^{\beta+1}$ and $HD^\alpha V^\beta = D^{\alpha-1}V^\beta$, which we deduce from the commutation relationships.

Thus after 1995 moves we have $D^\alpha V^\beta x_0 = x_0$ where $\beta \in \{0,1\}$ and $|\alpha| + \beta \leq 1995$. We consider two cases:-

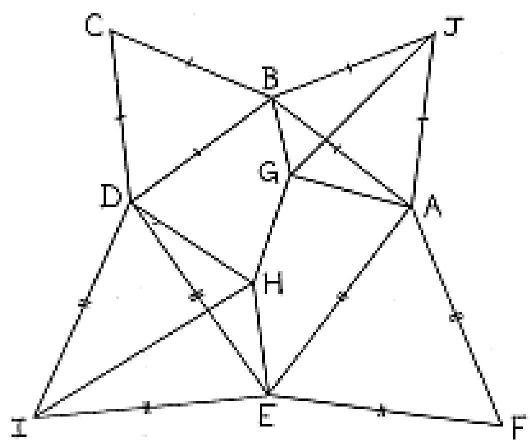
- $\beta = 0$. Then $D^\alpha x_0 = x_0$, and so $2^\alpha x_0 = x_0$, or $x_0 = 1$.
- $\beta = 1$. Then $D^\alpha \frac{1}{x_0} = x_0$, and so $2^\alpha = x_0^2$, or $2^{\alpha/2} = x_0$.

x_0 is maximised when α is maximised, and $\alpha \leq 1994$ by assumption. Thus $\alpha = 1994$ for maximisation, and $x_0 = 2^{997}$ is the maximum possible value.

Now $VH^{1994}2^{997} = 2^{997}$, and so this maximum is attained.

5. **Solution of Mark Berman:**

We construct equilateral triangles AJB and EID externally on the hexagon.



This construction can be motivated by noticing that such a construction creates symmetry about BE (since $\triangle BCD$, $\triangle AEF$ are equilateral, and $ABDE$ is a kite) and that it also creates two cyclic quadrilaterals: $AGBJ$ and $EHDI$.

Applying Ptolemy's theorem[13] in $AGBJ$, we get $AG \cdot JB + GB \cdot AJ = JG \cdot AB$, and so $AG + GB = JG$, since $AB = AJ = JB$. Similarly in $EHDI$ we get $HE + HD = HI$.

Now by the triangle inequality we get that $JG + GH + HI \geq JI = CF$, and so $AG + GB + GH + DH + HE \geq CF$.

Note:

There is equality if and only if J, G, H, I are collinear.

The condition that $\angle AGB = \angle EHD = 120^\circ$ is unnecessary. In fact, if G and H are any points in the plane, then Ptolemy's theorem becomes the Ptolemy-Euler inequality $AG \cdot JB + GB \cdot AJ \geq JG \cdot AB$, and similarly for H. The result still follows.

[13] See *Mathematical Digest*, no. 99, April 1995, pages 24-25.

6. **Solution of Elitca Mitova:**

We split the given sets into the two sets $\{1, 2, \ldots, p\}$ and $\{p+1, p+2, \ldots, 2p\}$. Modulo p these sets are the same, namely, a complete residue system modulo p.

Thus the problem is equivalent to choosing, for $0 \leq q \leq p$ and $0 \leq r \leq p-1$, q elements from $\{0, 1, \ldots, p-1\}$ whose sum is $\equiv_p r$ and then $p-q$ elements from $\{0, 1, \ldots, p-1\}$ whose sum is $\equiv_p -r$. More specifically, let us denote by $R_\alpha(\beta)$ the number of possible ways of choosing α elements from $\{0, 1, \ldots, p-1\}$ whose sum is $\equiv_p \beta$: the problem is then asking for the value of

$$S = \sum_{q=0}^{p} \sum_{r=0}^{p-1} R_q(r)\, R_{p-q}(-r).$$

However, determining $p-q$ elements from $\{0, 1, \ldots, p-1\}$ whose sum is $\equiv_p -r$ is equivalent to determining those q elements which will remain behind and whose sum is necessarily $\equiv_p r$, in other words, $R_q(r) = R_{p-q}(-r)$. Thus

$$S = \sum_{q=0}^{p} \sum_{r=0}^{p-1} R_q(r)^2.$$

The case where $q = 0$ or $q = p$ is trivial: there we have $R_q(0) = 1$ and $R_q(r) = 0$ for $1 \leq r \leq p-1$. Hence

$$S = 2 + \sum_{q=1}^{p-1} \sum_{r=0}^{p-1} R_q(r)^2.$$

So suppose $1 \leq q \leq p-1$ and $0 \leq r \leq p-1$. Then $\{p, 2p, \ldots, qp\}$ is a complete residue system modulo q (since $(p, q) = 1$) and so there is some unique index i with $1 \leq i < q$ such that $ip \equiv_q -r$. Then $\frac{ip+r}{q}$ is an integer. Given a q element subset of $\{0, 1, \ldots, p-1\}$ whose sum is $\equiv_p r$, we can subtract $\frac{ip+r}{q}$ from each element and get a q element subset of $\{0, 1, \ldots, p-1\}$ whose sum is $\equiv_p 0$. This operation is invertible, because given a set whose sum is $\equiv_p 0$ we can add $\frac{ip+r}{q}$ to each element and get a set whose sum is $\equiv_p r$.

In other words we have a bijection, and $R_q(r) = R_q(0)$. Hence

$$S = 2 + p\sum_{q=1}^{p-1} R_q(0)^2.$$

Furthermore,

$$R_q(0) + R_q(1) + \cdots + R_q(p-1) = \binom{p}{q}$$

and so $p \cdot R_q(0) = \binom{p}{q}$. Hence

$$\begin{aligned} S &= 2 + p\sum_{q=1}^{p-1}\left(\frac{\binom{p}{q}}{p}\right)^2 \\ &= 2 + \frac{1}{p}\sum_{q=1}^{p-1}\binom{p}{q}^2 \\ &= 2 + \frac{1}{p}\left[\binom{2p}{p} - 2\right]. \end{aligned}$$

Here we use the Chu-Vandermonde identity $\sum_{q=0}^{p}\binom{p}{q}^2 = \binom{2p}{p}$.

The results of the South African team:-

Team member	Q1	Q2	Q3	Q4	Q5	Q6	$\sum$
Mark Berman	7	0	3	7	7	0	24
David Fraser	7	1	4	6	0	1	19
David Hatton	7	0	0	7	0	0	14
Elitca Mitova	7	0	1	1	2	7	18
Andrew Skeen	7	0	1	-	-	-	8
Jan van Zyl Smit	7	0	1	2	2	0	12
Average	7	0.14	1.67	4.6	2.2	1.6	
Competition average	5.06	1.71	3.13	4.60	3.42	1.06	

Andrew Skeen was absent (due to illness) from the second day of competition.

Mark Berman and David Fraser received Bronze Medals, while all other members of the team received an Honourable Mention. The qualifying score for a Gold Medal was 37 points, for a Silver Medal 29 points, and for a Bronze Medal 19 points. An Honourable Mention is awarded to those participants who gain no medal but solve at least one problem completely.

In the (unofficial) team standings, South Africa placed 41^{st} out of 73 participating teams, with a total of 95 points.

The Way Forward

The Old Mutual, South Africa's largest life assurance society, will sponsor the South African Mathematical Society's International Mathematical Olympiad programme for the next two years.

An important feature of Old Mutual's sponsorhip is that substantial efforts will be made by the South African Mathematical Society to ensure that everybody in South Africa will be made aware of the Talent Search and encouraged to take part. Moreover, financial support will be made available for the launching of projects to increase participation in regional competitions and Olympiads and run programmes to identify and develop high school students who show mathematical promise.

Old Mutual has established itself as a leader in supporting mathematics education in South Africa across a broad spectrum of activities. Projects which Old Mutual supports include MATHS 24 for primary schools, the national Mathematics and Computer Science Olympiads, and *Mathematical Digest*. In addition, Old Mutual is a major supporter of AMESA, the Association for Mathematics Education of South Africa.

The South African Mathematical Society (SAMS) consists of mathematicians who work in Universities, Technikons, colleges, research institutes, commerce and industry. The Society is committed to the development of mathematics in all its aspects for all the people of South Africa. The Society sees the Talent Search and IMO programme as an important means of promoting the image of mathematics and attracting more students into mathematical careers in research, teaching and applications in industry and commerce.

Old Mutual's commitment to the improvement of mathematics education in South Africa is deeply appreciated.

John Webb
Convener, South African Committee
for the International Mathematical Olympiad

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